Instruction [*Arahan*]:
Answer ALL questions. Each question carries equal weight. Total 3×100 marks = 300 marks.
[*Jawab semua soalan. Setiap soalan mempunyai pemberat yang sama. Jumlah markah =* $3\times 100 = 300$]

1. Consider a 3-dimensional box containing N non-interacting ideal gas particle at a constant temperature T. Let $n(\mathbf{r})$ denotes the particle density in a small volume dV around the point $\mathbf{r}$, and $d\tau \equiv dv_x dv_y dv_z$ denotes the volume element in velocity space around $\mathbf{v}$.

[*Pertimbangkan suatu kotak bertiga dimensi yang mengandungi N zarah unggul tidak bersaling-tindak pada suhu T. Biar* $n(\mathbf{r})$*mewakili ketumpatan zarah dalam isipadu kecil dV di sekitar titik* $\mathbf{r}$*, dan* $d\tau \equiv dv_x dv_y dv_z$ *mewakili unsur isipadu dalam ruang halaju di sekitar* $\mathbf{v}$*.*]

(a) Let $P(\mathbf{r})dV$ be the probability that a particle will be in volume dV around $\mathbf{r}$, express $P(\mathbf{r})dV$ in terms of N, $n(\mathbf{r})$ and dV.

[*Biar* $P(\mathbf{r})dV$ *mewakili kebarangkalian suatu zarah berada dalam isipadu dV di sekitar* $\mathbf{r}$*, nyatakan* $P(\mathbf{r})dV$ *dalam sebutan N,* $n(\mathbf{r})$ *dan dV.*]

[15 marks]

(b) Denoting the probability for a particle to have a velocity $\mathbf{v}$ in that velocity volume by $f(\mathbf{v})d\tau$, express $dN(\mathbf{r},\mathbf{v})$, the number of particles with a velocity in a region $d\tau$ around the velocity $\mathbf{v}$, that are located in a volume dV around the location $\mathbf{r}$.

[*Andaikan* $f(\mathbf{v})d\tau$ *mewakili kebarangkalian suatu zarah berhalaju* $\mathbf{v}$ *berada dalam isipadu halaju berkenaan, nyatakan* $dN(\mathbf{r},\mathbf{v})$*, bilangan zarah berhalaju dalam rantau* $d\tau$ *di sekitar halaju* $\mathbf{v}$*, yang berada dalam isipadu dV di sekitar kedudukan* $\mathbf{r}$*.*]

[15 marks]

(c) What is the dimension for $P(\mathbf{r})$?
[*Apakah dimensi* $P(\mathbf{r})$?]

[15 marks]

(d) What is the dimension for $f(\mathbf{v})$?
[*Apakah dimensi* $f(\mathbf{v})$?]

[15 marks]

(e) $f(\mathbf{v})$ defined above is the probability per unit velocity volume for a particle to have a velocity near $\mathbf{v}$. The probability function is known as the Maxwell distribution. Derive explicitly, augmented with all necessary mathematical arguments, that the Maxwell distribution takes the form $f(\mathbf{v})=Ce^{-Av^2}$, where A and C are constants. Note: You do not need to derive what A and C are.

[$f(\mathbf{v})$ *yang didefinasikan di atas merupakan kebarangkalian per unit isipadu halaju bagi suatu zarah yang berhalaju* $\mathbf{v}$*. Fungsi kebarangkalian tersebut dikenali sebagai taburan Maxwell. Terbitkan secara eksplisit, disokongkan dengan hujah-hujah matematik yang perlu, bahawa taburan Maxwell mengambil bentuk* $f(\mathbf{v})=Ce^{-Av^2}$ *, di mana A dan C adalah pemalar. Nota:*

Anda tidak perlu menerbitkan A dan C.]

[40 marks]

2. The average energy at inverse temperature $\beta = 1/kT$ of a non-interacting spin-1/2 paramagnetic system is $\langle E \rangle = Z^{-1} \sum_{\{\sigma\}} E(\sigma_1, \sigma_2, \cdots, \sigma_n) e^{-\beta E(\sigma_1, \sigma_2, \cdots, \sigma_n)}$, with probabilities $P = Z^{-1} e^{-\beta E}$; Z is the partition function for the system. Assume there is a total of n magnetic moment in the system. With H denoting the external magnetic field, the partition function of the system is given by $Z(\beta, H) = [z(\beta, H)]^n$, $z(\beta, H) = 2\cosh(\beta \mu_B H)$.

[*Tenaga purata pada suhu songsang* $\beta = 1/kT$ *sistem zarah paramagnet tidak saling-bertindak spin-1/2 adalah* $\langle E \rangle = Z^{-1} \sum_{\{\sigma\}} E(\sigma_1, \sigma_2, \cdots, \sigma_n) e^{-\beta E(\sigma_1, \sigma_2, \cdots, \sigma_n)}$, *dengan kebarangkalian-kebarangkalian* $P = Z^{-1} e^{-\beta E}$; *Z adalah fungsi partisi sistem. Andaikan terdapat n momen magnet dalam sistem ini. Fungsi partisi sistem adalah diberi oleh* $Z(\beta, H) = [z(\beta, H)]^n$, $z(\beta, H) = 2\cosh(\beta \mu_B H)$, *dengan H mewakili medan magnet luar.*]

(a) Prove that the average energy can be expressed as $\langle E \rangle = -\dfrac{\partial \ln Z}{\partial \beta}$.

[*Buktikan bahawa min tenaga boleh dinyatakan sebagai* $\langle E \rangle = -\dfrac{\partial \ln Z}{\partial \beta}$]

[20 marks]

(b) Write down the average magnetization of the system, $\langle M \rangle$ in terms of the partition function Z and H.

[*Tuliskan min kemagnetan sistem,* $\langle M \rangle$, *dalam sebutan fungsi partisi Z dan H.*]

[10 makrs]

(c) What is the relation between $\langle M \rangle$ and the average projection of a spin, $\langle \sigma \rangle$? [*Apakah hubungan di antara* $\langle M \rangle$ *dan* $\langle \sigma \rangle$, *min projeksi spin tunggal?*]

[10 marks]

(d) Let $x = \beta \mu_B H$, sketch the graph of $\langle \sigma \rangle$ vs. x. State clearly the lower and upper limits of $\langle \sigma \rangle$ in the graph.

[*Biar* $x = \beta \mu_B H$, *lakarkan graf* $\langle \sigma \rangle$ *lwn. x. Nyatakan dengan jelas limit bawah dan limit atas* $\langle \sigma \rangle$ *dalam graf anda.*]

[20 marks]

(e) What is the limit of $\langle\sigma\rangle$ in the limit $|x| \rightarrow 0$? Provide a physical interpretation to this behavior.
[*Apakah limit* $\langle\sigma\rangle$ *dalam limit* $|x| \rightarrow 0$? *Berikan suatu interpretasi fizikal kepada kelakuan tersebut.*]

[20 marks]

(f) What is the limit of $\langle\sigma\rangle$ in the limit $|x| \rightarrow \pm\infty$. Provide a physical interpretation to this behavior.
[*Apakah limit* $\langle\sigma\rangle$ *dalam limit* $|x| \rightarrow \pm\infty$? *Berikan suatu interpretasi fizikal kepada kelakuan tersebut.*]

[20 marks]

3. Consider a classical mechanical system comprised of only a particle with mass m that can only move along the x-axis, and is subjected to a potential field, U(x). Let the kinetic energy of the particle be denoted by K.
[*Pertimbangkan suatu sistem mekanik klasik terdiri daripada satu zarah berjisim m yang hanya bergerak sepanjang paksi-x, dan tertakluk kepada suatu medan keupayaan U(x). Biar tenaga kinetik zarah diwakili oleh K.*]

(a) Write down the total energy of the system in terms of U and K.
[*Tuliskan tenaga jumlah sistem dalam sebutan U dan K.*]

[10 marks]

(b) Write down the kinetic energy of the system in terms of the momentum of the particle, p_x.
[*Tuliskan tenaga kinetik sistem dalam sebutan momentum zarah* p_x.]

[10 marks]

(c) Write down explicitly the expression for the single particle partition function, z_c, in terms of momentum and potential energy at the inverse temperature $\beta = 1/kT$.
[*Tuliskan secara eksplisit ungkapan untuk fungsi partisi zarah tungal,* z_c, *dalam sebutan-sebutan momentum dan tenaga keupayaan pada suhu songsang* $\beta = 1/kT$.]

[20 marks]

(d) What is $P(p_x, x)$, the probability density of finding a particle in a state $\{x, p_x\}$?
[*Apakah* $P(p_x, x)$, *ketumpatan kebarangkalian untuk menemui suatu zarah dalam keadaan* $\{x, p_x\}$?]

[20 marks]

(e) The average value of a physical observable A is $\langle A \rangle = \int A(x, p_x)P(x, p_x)dx\ dp_x$. What is the expected value of the potential energy, $\langle U \rangle$?
[*Nilai min suatu pencerap fizikal A adalah* $\langle A \rangle = \int A(x, p_x)P(x, p_x)dx\ dp_x$. *Apakah nilai jangkaan tenaga keupayaan* $\langle U \rangle$ *?*]

[20 marks]

(f) What is the expected value of the kinetic energy, $\langle K \rangle$?
[*Apakah nilai jangkaan tenaga kinetik* $\langle K \rangle$?]

[20 marks]

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ZCT 207 Statistical Mechanics
KSCP Solution

Q1.
Solution: Page 23, Amit, eq. (1.1.33)

1(a) $P(\mathbf{r})\mathrm{d}V = \frac{1}{N} n(r) dV$ [15/100]

Solution: Page 25, Amit, eq. (1.1.35)

1(b) $dN(\mathbf{r},\mathbf{v}) = n(\mathbf{r})\mathrm{d}V f(\mathbf{v})\mathrm{d}\tau = N\, P(\mathbf{r})\, f(\mathbf{v})\, \mathrm{d}\tau\, \mathrm{d}V$ [15/100]

Solution: Page 25, Amit,

1(c) $[\mathrm{P}(\mathbf{r})] = \mathrm{L}^{-3}$ [15/100]

1(d) $[\mathrm{f}(\mathbf{v})] = \mathrm{L}^{-3}\,\mathrm{T}^{3}$ [15/100]

1(e) Page 25 – 27, Amit. [40/100]

Q2.
Solution:

2(a) $\langle E \rangle = Z^{-1} \sum_{\{\sigma\}} E e^{-\beta E} = -Z^{-1} \sum_{\{\sigma\}} \frac{\partial}{\partial \beta}\left(e^{-\beta E}\right) = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial \beta}$ [20/100]

(b) $\langle M \rangle = \frac{1}{\beta}\frac{\partial \ln Z}{\partial H}$ [10/100]

(c) $\langle M \rangle = n\langle \sigma \rangle$ [10/100]

(d) page 170 Amit, Fig. 2.5.1. [20/100]

(e) $\langle \sigma \rangle$ = 0 in the limit $|x| \to 0$. Physical interpretation see page 170. Either interpretation (a) or (b) is accepted as full answer. [20/100]

(f) $\langle \sigma \rangle$ = ±1 in the limit $|x| \to \pm\infty$. Physical interpretation see page 170. Either interpretation (a) or (b) is accepted as full answer. [20/100]

Q3.

Solution:

(a) $E = U + K$ [10/100]

(b) $K = p_x/2m$ [10/100]

(c) $z_c = \int e^{-\beta\left[\frac{p_x^2}{2m}+U(x)\right]} dx dp_x$ [20/100]

(d) Page 266, Amit, Eq. (3.4.22). $P(p_x, x) = z_c^{-1} \exp\left\{-\beta\left[\frac{p_x^2}{2m}+U(x)\right]\right\}$ [20/100]

(e)

$$\langle U\rangle = \frac{\int e^{-\beta(\frac{p_x^2}{2m})} dp_x \cdot \int U(x) e^{-\beta U(x)} dx}{\int e^{-\beta U(x)} dx \cdot \int e^{-\beta(\frac{p_x^2}{2m})} dp_x} = \frac{\int U(x) e^{-\beta U(x)} dx}{\int e^{-\beta U(x)} dx} =$$
$$-\frac{\partial}{\partial\beta} \ln z_U;\ z_u = \int e^{-\beta U(x)} dx$$

[20/100]

(f)

$$\langle K\rangle = \frac{\int e^{-\beta U(x)} dx \cdot \int \frac{p_x^2}{2m} e^{-\beta(\frac{p_x^2}{2m})} dp_x}{\int e^{-\beta U(x)} dx \cdot \int e^{-\beta(\frac{p_x^2}{2m})} dp_x} = \frac{\int \frac{p_x^2}{2m} e^{-\beta(\frac{p_x^2}{2m})} dp_x}{\int e^{-\beta(\frac{p_x^2}{2m})} dp_x} =$$
$$-\frac{\partial}{\partial\beta} \ln\left(\int e^{-\beta(\frac{p_x^2}{2m})} dp_x\right) = -\frac{\partial}{\partial\beta} \ln z_K$$

[20/100]