ZCA 207 Statistical Mechanics
Test II
`Tuesday, 30 Mar 10, 4.00 pm — 5.00 pm`
`Venue : SK1`

This test contains only ONE question printed on two pages. Answer all parts.

Name:

Matrix No.

1. By definition, the 1-particle partition function for a given particle labelled is i is $z_i = \sum_{\text{all microstates}} e^{-\beta\epsilon(n)}$, where $\beta = \frac{1}{kT}$ is the inverse temperature parameter, $\epsilon(n)$ is the energy correspond to a microstate n.

   (a) Write down the energy levels of an one-dimensional harmonic oscillator oscillating at an intrinsic angular frequency ω. You must define clearly the quantity you write down.

   [10 marks]

   (b) Referring to your answer in (1a), what is the 1-particle partition function of a quantum oscillator? Simplify your expression by summing over all the energy states n.

   [20 marks]

   (c) Based on the partition function of the 1-particle partition function as obtained in (1b), deduce the degree of excitation, $\langle n \rangle$ (the Bose-Einstein distribution) for a given quantum oscillator.

   [20 marks]

(d) Deduce the average energy of a single oscillator, $\langle \epsilon \rangle$.

[20 marks]

(e) From (1d), deduce the heat capacity of a single harmonic oscillator as a function of temperature T.

[20 marks]

(f) Deduce the limit of the result in (1e) when $kT \gg \hbar\omega$.

[10 marks]

Solutions

1. (a) The energy levels of a QHO is given by

$$\epsilon(n) = \epsilon_n = (\frac{1}{2} + n)\hbar\omega,$$

where $n = 0, 1, 2, \cdots$. The quantum number n characterises the degree of excitation of the QHO.

(b) see page 246 Amit, Eq.(3.2.7)

$$z = \sum_{n=0}^{n=\infty} e^{-\beta\epsilon_n} = \sum_{n=0}^{n=\infty} e^{-\beta(\frac{1}{2}+n)\hbar\omega} = e^{-\beta\frac{\hbar\omega}{2}} \sum_{n=0}^{n=\infty} \exp(-\beta n\hbar\omega) = \frac{e^{-\beta\frac{\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}$$

(c) see page 302, Amit.

$$\langle n \rangle = z^{-1} \sum_{n=0}^{n=\infty} n e^{-\beta(\frac{1}{2}+n\hbar\omega)} = \frac{\sum_{n=0}^{n=\infty} n e^{-\beta\hbar\omega}}{\sum_{n=0}^{n=\infty} e^{-\beta\hbar\omega}} = -\frac{d}{dx}\ln\left(\frac{1}{1 - e^{-\beta\hbar\omega}}\right) = \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega} - 1}.$$

(d) see page 302, Amit.

$$\langle \epsilon \rangle = -\frac{\partial}{\partial\beta}\ln z = -\frac{\partial}{\partial\beta}\ln\frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega/2}} = -\frac{\partial}{\partial\beta}[-\beta\hbar\omega/2 - \ln(1 - e^{-\beta\hbar\omega/2})] = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

(e)

$$c = \frac{\partial\langle\epsilon\rangle}{\partial T} = \frac{\partial}{\partial T}\left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT-1}}\right) = k\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}} - 1)^2} \tag{1}$$

(f) Let $x = \beta\hbar\omega$. In the high temperature limit, $x \to 0$, $e^{\beta\hbar\omega} = e^x \approx 1+x$, hence $c \approx kx^2\frac{(1+x)}{[(1-x)-1]^2} = k(1+x) = k$.