Lecture Notes
Vector Analysis
ZCT 211

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Triple cross product

3. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

4. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

   $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$

the associative law for vector cross products is not valid for all vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

47. Prove: (*a*) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

(*b*) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$.

- LHS of (a)

$$(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

- RHS of (a)
- Expand fully the expressions

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

The compare component by components on both sides, and the results follow.
The trick is: There is no trick involved. Just expand both sides and compare.

The proof of (b)

know → $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Show this → (*b*) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$.

(b) can be trivially proofed based on the (a)

- Simply swap the order of (**A** x **B**) with **C** in (a) (hence introducing a minus sign) and then rename **B** -> **A**, **C** -> **B**, **A**->**C** in the resultant formula:

$$(A \times B) \times C =$$

$$-A \times (B \times C) = -B(A \cdot C) + C(A \cdot B)$$

-

↓ **B** -> **A**, **C** -> **B**, **A**->**C**

$$-C \times (A \times B) = -A(C \cdot B) + B(C \cdot A)$$

$$(A \times B) \times C = B(A \cdot C) - A(B \cdot C)$$

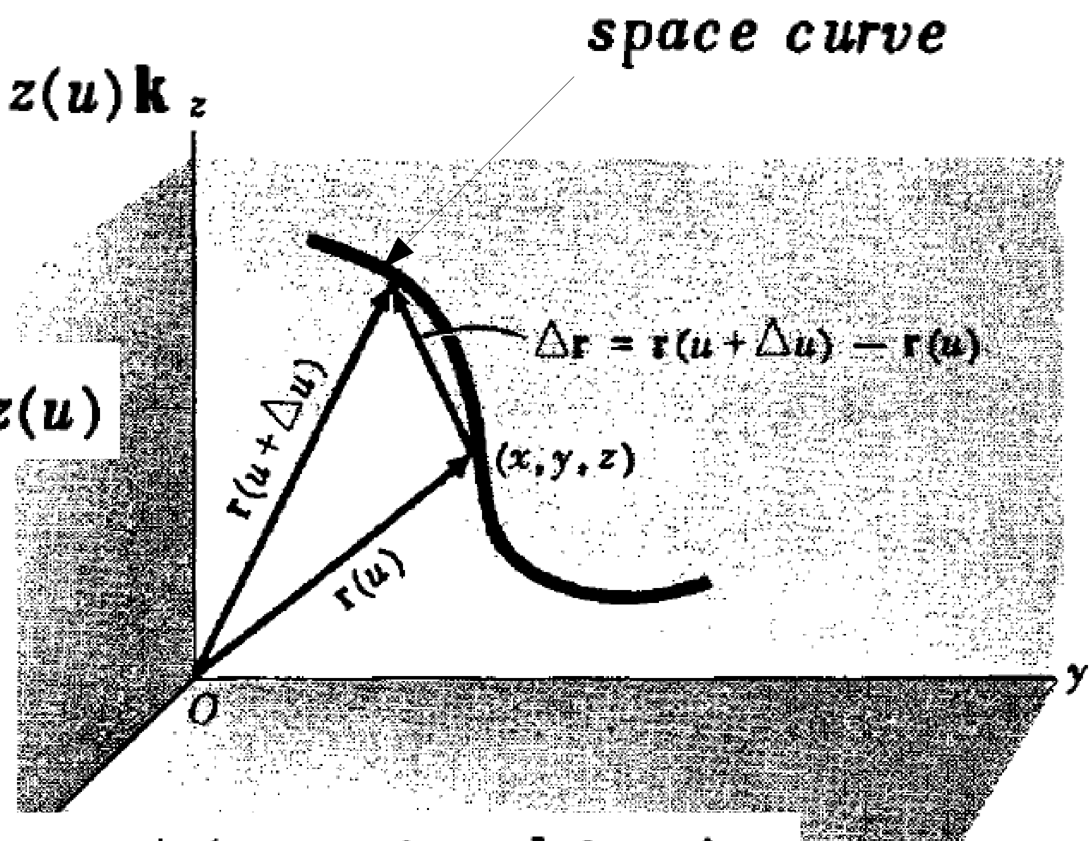
Vector Differentiation

$$\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$$

parametric equations

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

$$\lim_{\Delta u \to 0} \frac{\Delta \mathbf{r}}{\Delta u} = \frac{d\mathbf{r}}{du}$$



a vector in the direction of the tangent to the space curve at (x, y, z) and is given by

$$\frac{d\mathbf{r}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

If $u = t$, then $d\mathbf{r}/dt = \mathbf{v}$

DIFFERENTIATION FORMULAS.

If **A**, **B** and **C** are differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u, then

1. $$\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$$

2. $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

3. $$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

4. $$\frac{d}{du}(\phi\mathbf{A}) = \phi\frac{d\mathbf{A}}{du} + \frac{d\phi}{du}\mathbf{A}$$

5. $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}\times\frac{d\mathbf{C}}{du} + \mathbf{A}\cdot\frac{d\mathbf{B}}{du}\times\mathbf{C} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}\times\mathbf{C}$$

6. $$\frac{d}{du}\{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\} = \mathbf{A}\times(\mathbf{B}\times\frac{d\mathbf{C}}{du}) + \mathbf{A}\times(\frac{d\mathbf{B}}{du}\times\mathbf{C}) + \frac{d\mathbf{A}}{du}\times(\mathbf{B}\times\mathbf{C}$$

Can you derive these equations using differentiation by parts?

Vector as a function of scalar or a set of scalar

$$\boldsymbol{A} = A_i \boldsymbol{i} + A_j \boldsymbol{j} + A_k \boldsymbol{k}$$
$$A_i = A_i(u)$$
$$A_i = A_i(x, y, z)$$
$$A_i = A_i(x, y, z, t)$$

$$\boldsymbol{A} = \boldsymbol{A}(u)$$
$$\boldsymbol{A} = \boldsymbol{A}(x, y, z)$$
$$\boldsymbol{A} = \boldsymbol{A}(x, y, z, t)$$

Topics

1. Calculations involving algebra or the taking the derivations of vectors
2. Application on kinematics (involving **r**, **v**, **a**)
3. Application on dynamics / rotational dynamics, **F** = *m***a** etc.
4. Differential geometry

$$\boldsymbol{A} = A_i \boldsymbol{i} + A_j \boldsymbol{j} + A_i \boldsymbol{j}$$

$$\boldsymbol{A} = \boldsymbol{A}(u)$$

$$\boldsymbol{A} = \boldsymbol{A}(x, y, z)$$

$$\boldsymbol{A} = \boldsymbol{A}(x, y, z, t)$$

Partial derivative of a function of multiple variable

$$\frac{\partial}{\partial x} f(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z)}{(\Delta x)}$$

$$\frac{\partial}{\partial y} f(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y, z)}{(\Delta y)}$$

$$\frac{\partial}{\partial z} f(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z)}{(\Delta z)}$$

Example of partial derivative

$$f(x, y, z) = x^2 y^3 z^4$$

$$\frac{\partial}{\partial x} f(x, y, z) = 2x(y^3 z^4)$$
$$\frac{\partial}{\partial y} f(x, y, z) = 3y^2(x^2 z^4)$$
$$\frac{\partial}{\partial z} f(x, y, z) = 4z^3(x^2 y^3)$$

PARTIAL DERIVATIVES OF VECTORS

If $\mathbf{A}$ is a vector depending on more than one scalar variable, say x, y, z for example, then we write $\mathbf{A} = \mathbf{A}(x, y, z)$.

$$\frac{\partial \mathbf{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{A}(x+\Delta x, y, z) - \mathbf{A}(x,y,z)}{\Delta x}$$

$$\frac{\partial \mathbf{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\mathbf{A}(x, y+\Delta y, z) - \mathbf{A}(x,y,z)}{\Delta y}$$

$$\frac{\partial \mathbf{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\mathbf{A}(x,y, z+\Delta z) - \mathbf{A}(x,y,z)}{\Delta z}$$

We shall assume all vectors encountered are differentiable to any order n needed

An example would be $\mathbf{A}(x,y,z)$ as vector potential at fixed time.

Differential of a function of single and multi-variables

Recall your ZCA 110:

$$df(x) = \frac{df(x)}{dx} dx$$

For single variable function

$$df(x, y, z) = \frac{\partial f(x, y, z)}{\partial x} dx + \frac{\partial f(x, y, z)}{\partial y} dy + \frac{\partial f(x, y, z)}{\partial z} dz$$

For multiple variables function

DIFFERENTIALS OF VECTORS

1. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, then $d\mathbf{A} = dA_1\mathbf{i} + dA_2\mathbf{j} + dA_3\mathbf{k}$

2. $d(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot d\mathbf{B} + d\mathbf{A}\cdot\mathbf{B}$

3. $d(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times d\mathbf{B} + d\mathbf{A}\times\mathbf{B}$

4. If $\mathbf{A} = \mathbf{A}(x,y,z)$, then $d\mathbf{A} = \frac{\partial\mathbf{A}}{\partial x}dx + \frac{\partial\mathbf{A}}{\partial y}dy + \frac{\partial\mathbf{A}}{\partial z}dz$, etc.

Higher derivatives can be defined as in the calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{A}}{\partial z}\right)$$

$$\frac{\partial^2 \mathbf{A}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \qquad \frac{\partial^3 \mathbf{A}}{\partial x\, \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \mathbf{A}}{\partial z^2}\right)$$

1. $$\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}$$

2. $$\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\times\mathbf{B}$$

3. $$\frac{\partial^2}{\partial y\, \partial x}(\mathbf{A}\cdot\mathbf{B}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}\right\}$$

$$= \mathbf{A}\cdot\frac{\partial^2 \mathbf{B}}{\partial y\, \partial x} + \frac{\partial \mathbf{A}}{\partial y}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x}\cdot\mathbf{B}, \qquad \text{etc.}$$

2. Given $\mathbf{R} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, find

$$(a)\ \frac{d\mathbf{R}}{dt},\quad (b)\ \frac{d^2\mathbf{R}}{dt^2},\quad (c)\ \left|\frac{d\mathbf{R}}{dt}\right|,\quad (d)\ \left|\frac{d^2\mathbf{R}}{dt^2}\right|$$

$$(a)\ \frac{d\mathbf{R}}{dt} = \frac{d}{dt}(\sin t)\mathbf{i} + \frac{d}{dt}(\cos t)\mathbf{j} + \frac{d}{dt}(t)\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}$$

$$(b)\ \frac{d^2\mathbf{R}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{R}}{dt}\right) = \frac{d}{dt}(\cos t)\mathbf{i} - \frac{d}{dt}(\sin t)\mathbf{j} + \frac{d}{dt}(1)\mathbf{k} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$$

$$(c)\ \left|\frac{d\mathbf{R}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{2}$$

$$(d)\ \left|\frac{d^2\mathbf{R}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$$

6. (*a*) Find the unit tangent vector to any point on the curve

$$x = t^2+1, \quad y = 4t-3, \quad z = 2t^2-6t$$

(*a*) A tangent vector to the curve at any point is

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[(t^2+1)\mathbf{i} + (4t-3)\mathbf{j} + (2t^2-6t)\mathbf{k}\right]$$

$$= 2t\,\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}$$

The magnitude of the vector is $\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2}$

unit tangent vector is $\mathbf{T} = \dfrac{2t\,\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}}{\sqrt{(2t)^2 + (4)^2 + (4t-6)^2}}$

7. If **A** and **B** are differentiable functions of a scalar u, prove:

$$(a) \quad \frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

$$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \frac{d}{du}(A_1 B_1 + A_2 B_2 + A_3 B_3)$$

$$= (A_1\frac{dB_1}{du} + A_2\frac{dB_2}{du} + A_3\frac{dB_3}{du}) + (\frac{dA_1}{du}B_1 + \frac{dA_2}{du}B_2 + \frac{dA_3}{du}B_3)$$

$$= \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

Comment: solve by expanding **A** and **B** into their component form

A_i and B_i are functions of variable u, $A_i = A_i(u)$, $B_i = B_i(u)$

8. If $\mathbf{A} = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ and $\mathbf{B} = \sin t\ \mathbf{i} - \cos t\ \mathbf{j}$,

find (a) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$.

You can solve these questions by either (1) carry out the vector cross or dot product first, then take the derivative next, or (2) take the derivative of the bi-vectors first, then reduce the resultant vector derivatives.

$$\mathbf{A}\cdot\mathbf{B} = 5t^2\sin t - t\cos t$$

Method 1

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d}{dt}(5t^2\sin t - t\cos t)$$

$$= (5t^2 - 1)\cos t + 11t\sin t$$

Method 2

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{B}$$

$$= (5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k})\cdot(\cos t\ \mathbf{i} + \sin t\ \mathbf{j}) + (10t\,\mathbf{i} + \mathbf{j} - 3t^2\mathbf{k})\cdot(\sin t\ \mathbf{i} - \cos t\ \mathbf{j})$$

$$= (5t^2 - 1)\cos t + 11t\sin t$$

8. If $\mathbf{A} = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ and $\mathbf{B} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$,

$$(b)\quad \frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \hat{\mathbf{j}} & \mathbf{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= (t^3\sin t - 3t^2\cos t)\mathbf{i} - (t^3\cos t + 3t^2\sin t)\mathbf{j} + (5t^2\sin t - \sin t - 11t\cos t)\mathbf{k}$$

Visualise the curve of **C** = $\frac{d}{dt}(\mathbf{A}\times\mathbf{B})$ using Mathematica

12. Uniform circular motion

A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$ where ω is a constant. Show that (*a*) the velocity $\mathbf{v}$ of the particle is perpendicular to $\mathbf{r}$, (*b*) the acceleration $\mathbf{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin, (*c*) $\mathbf{r}\times\mathbf{v}$ = a constant vector.

(*a*) $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\omega\sin\omega t\,\mathbf{i} + \omega\cos\omega t\,\mathbf{j}$$

$$\mathbf{r}\cdot\mathbf{v} = [\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}]\cdot[-\omega\sin\omega t\,\mathbf{i} + \omega\cos\omega t\,\mathbf{j}]$$

(*b*) $$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = -\omega^2\cos\omega t\,\mathbf{i} - \omega^2\sin\omega t\,\mathbf{j}$$

$$= -\omega^2[\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}] = -\omega^2\mathbf{r}$$

The acceleration, directed toward the center of the circle, *centripetal acceleration.*

(c) $$\mathbf{r} \times \mathbf{v} = [\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}] \times [-\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix}$$

$$= \omega(\cos^2 \omega t + \sin^2 \omega t)\mathbf{k} = \omega \mathbf{k},$$ a constant vector.

This is the vector of angular momentum/*m*, fixed in direction and magnitude.
In other words, the angular speed is constant and the circular object is not making any angular acceleration.

Visualise uniform circular motion using Mathematica

14. Show that $\mathbf{A} \cdot \dfrac{d\mathbf{A}}{dt} = A\,\dfrac{dA}{dt}$.

$$\mathbf{A} \cdot \mathbf{A} = A^2, \quad \frac{d}{dt}(\mathbf{A} \cdot \mathbf{A}) = \frac{d}{dt}(A^2) = 2A\,\frac{dA}{dt}$$

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{A}) = \mathbf{A} \cdot \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{A} = 2A\,\frac{dA}{dt}$$

$$\mathbf{A} \cdot \frac{d\mathbf{A}}{dt} = A\,\frac{dA}{dt}$$

16. If $\phi(x,y,z) = xy^2z$ and $\mathbf{A} = xz\,\mathbf{i} - xy^2\,\mathbf{j} + yz^2\mathbf{k}$,

find $\dfrac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{A})$

$$\phi\mathbf{A} = (xy^2z)(xz\,\mathbf{i} - xy^2\,\mathbf{j} + yz^2\,\mathbf{k})$$

$$= x^2y^2z^2\,\mathbf{i} - x^2y^4z\,\mathbf{j} + xy^3z^3\,\mathbf{k}$$

$$\frac{\partial}{\partial z}(\phi\mathbf{A}) = \frac{\partial}{\partial z}(x^2y^2z^2\,\mathbf{i} - x^2y^4z\,\mathbf{j} + xy^3z^3\,\mathbf{k})$$

$$= 2x^2y^2z\,\mathbf{i} - x^2y^4\,\mathbf{j} + 3xy^3z^2\,\mathbf{k}$$

$$\frac{\partial^2}{\partial x\,\partial z}(\phi\mathbf{A}) = \frac{\partial}{\partial x}(2x^2y^2z\,\mathbf{i} - x^2y^4\,\mathbf{j} + 3xy^3z^2\,\mathbf{k})$$

$$= 4xy^2z\,\mathbf{i} - 2xy^4\,\mathbf{j} + 3y^3z^2\,\mathbf{k}$$

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi \mathbf{A}) = \frac{\partial}{\partial x}(4xy^2 z\ \mathbf{i} - 2xy^4\ \mathbf{j} + 3y^3 z^2\ \mathbf{k})$$

$$= 4y^2 z\ \mathbf{i} - 2y^4\ \mathbf{j}$$

28.

If $\mathbf{r}$ is the position vector of a particle of mass m relative to point O and $\mathbf{F}$ is the external force on the particle, then $\mathbf{r} \times \mathbf{F} = \mathbf{M}$ is the torque or moment of $\mathbf{F}$ about O. Show that $\mathbf{M} = d\mathbf{H}/dt$, where $\mathbf{H} = \mathbf{r} \times m\mathbf{v}$ and $\mathbf{v}$ is the velocity of the particle.

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v})$$

But
$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) + \frac{d\mathbf{r}}{dt} \times m\mathbf{v}$$

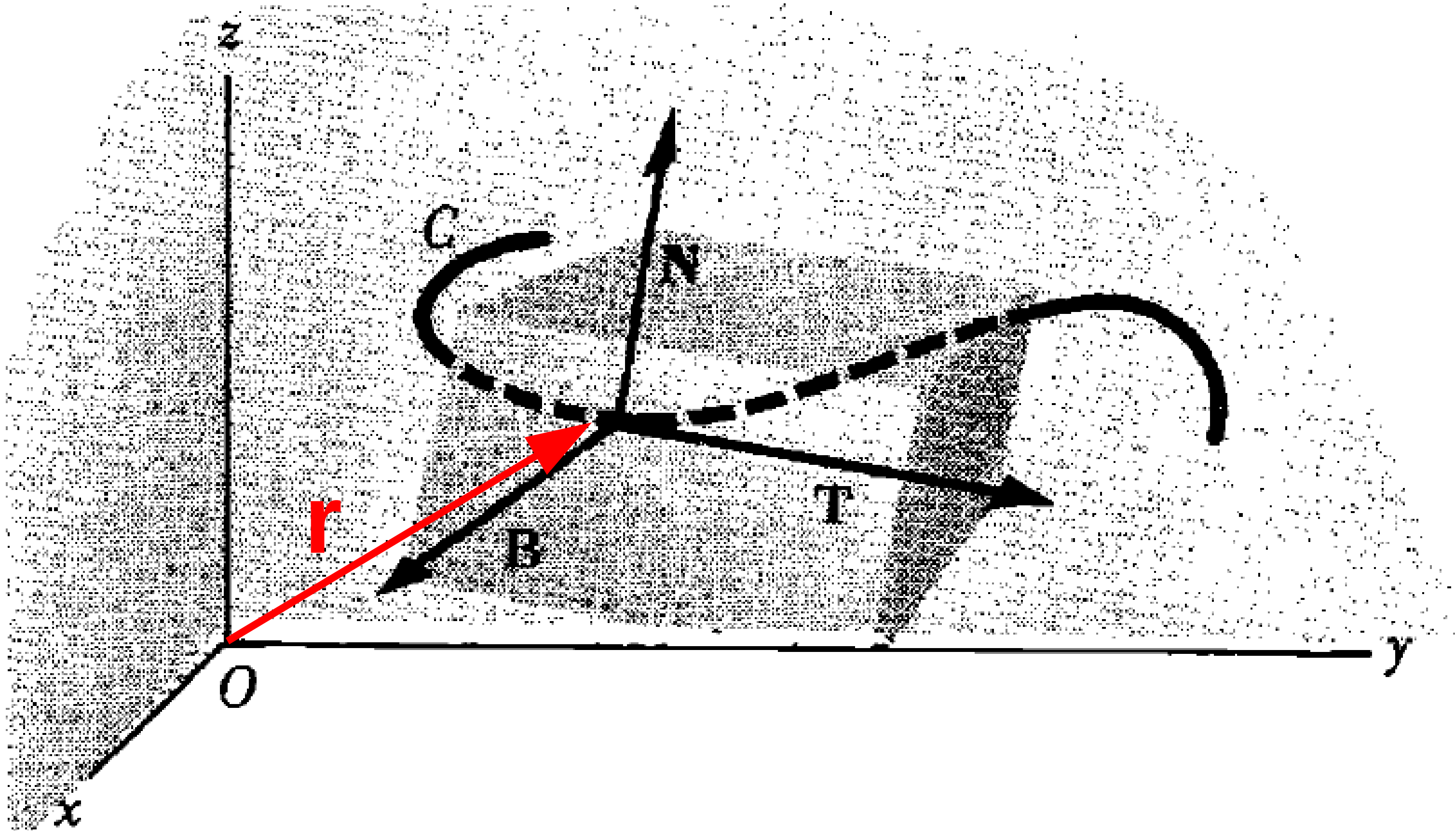
$$= \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) + \mathbf{v} \times m\mathbf{v}$$

$$= \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) + \mathbf{0}$$

$$\mathbf{M} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{H}}{dt}$$

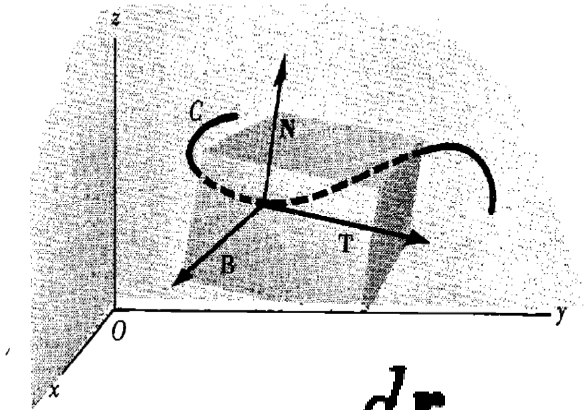
$\mathbf{H}$ is called the *angular momentum*.

Differential geometry



C is a space curve defined by the function **r**(u),

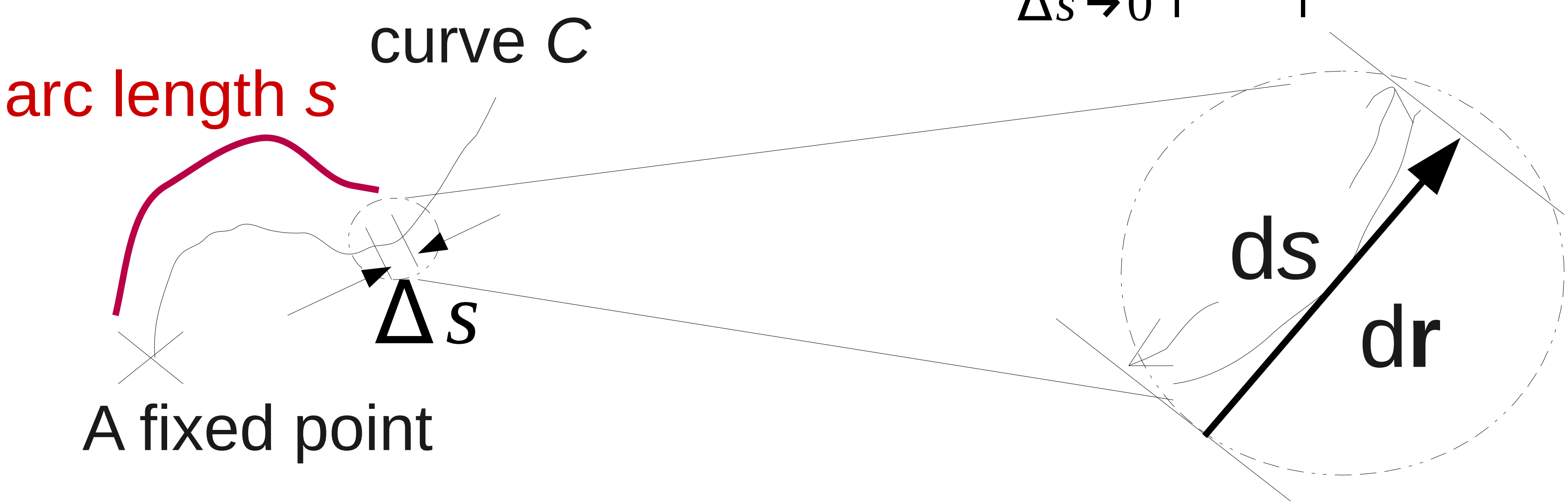
Unit tangent vector at a point *P* on a curve *C*



$\mathbf{T} = \frac{d\mathbf{r}}{ds}$ is a unit tangent vector to C

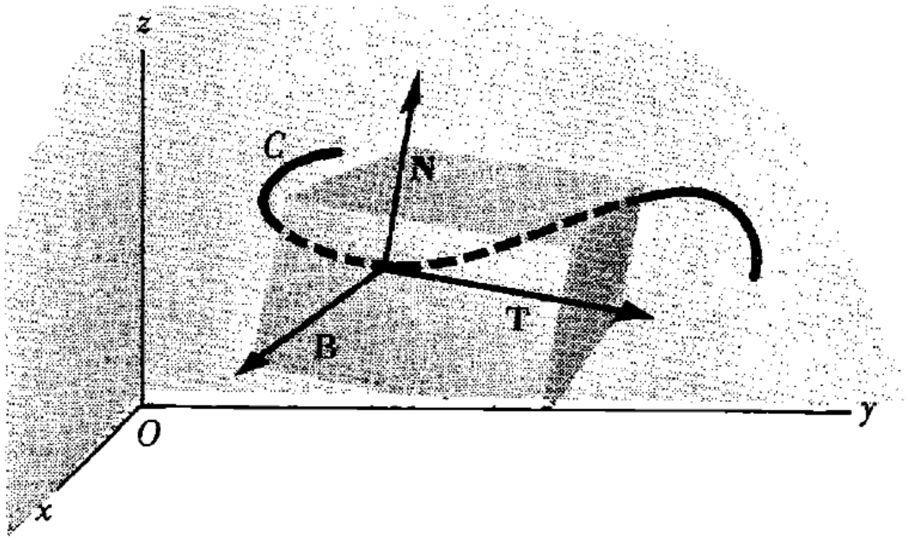
the arc length s measured from some fixed point on C

$$\lim_{\Delta s \to 0} |d\boldsymbol{r}| = ds$$



$$\frac{d\mathbf{T}}{ds}$$

The rate at which **T** changes with respect to the arc length *s* measures the curvature of C

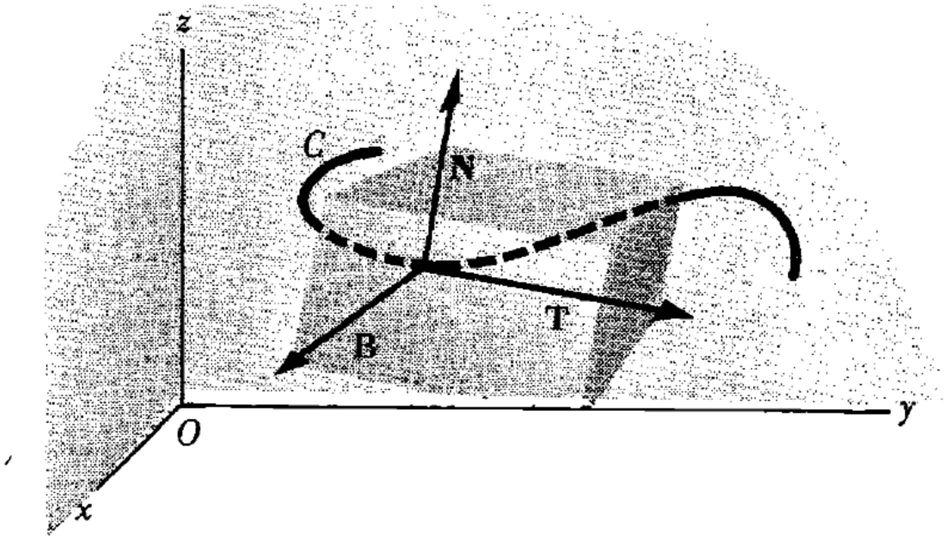


Te direction of $\frac{d\mathbf{T}}{ds}$ is $\mathbf{N}$, where $\mathbf{N}$ is normal to the curve at that point. $\mathbf{N}$ is also perpendicular to $\mathbf{T}$

Principle normal

We call the unit vector **N** (which is defined as a unit vector in the direction as that of $\frac{d\mathbf{T}}{ds}$)

principal normal



The magnitude of $\frac{d\mathbf{T}}{ds}$ at the point on the curve is denoted as $\left|\frac{d\mathbf{T}}{ds}\right| = \kappa$

Hence, we write

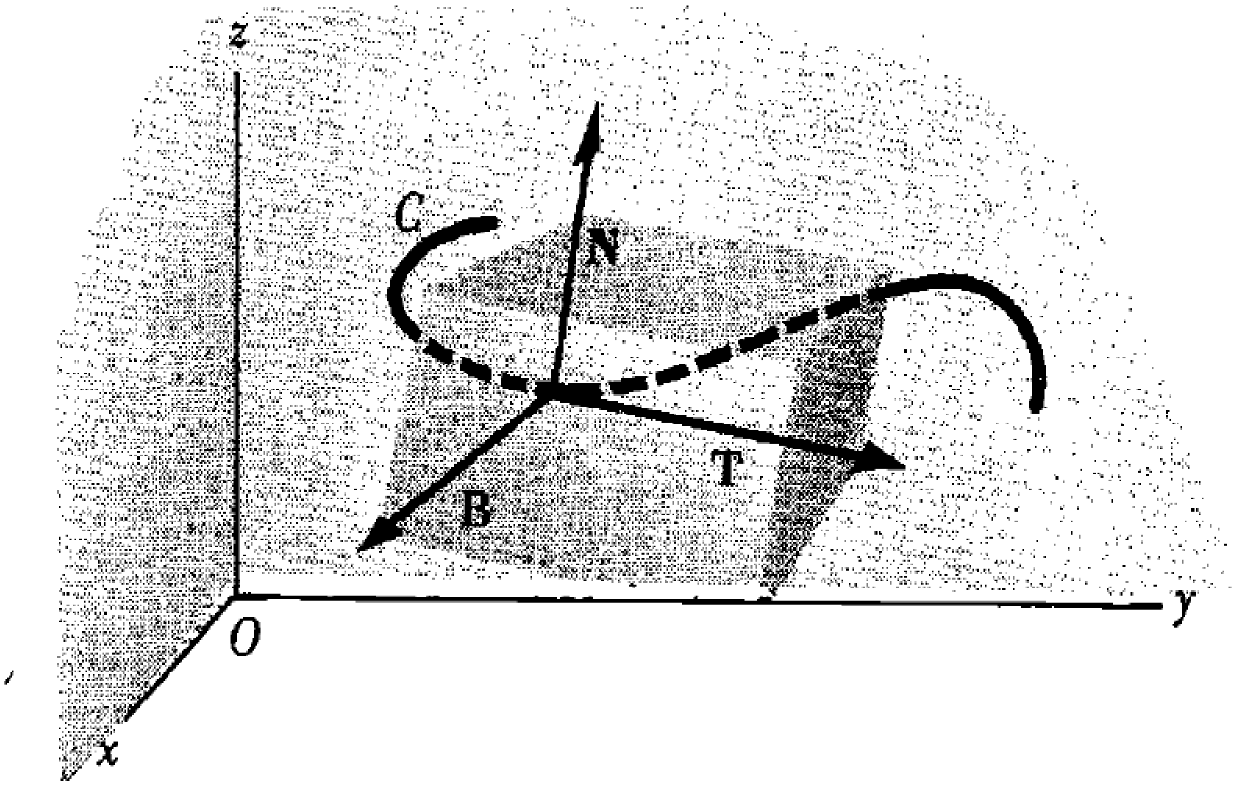
$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

κ is called the *curvature* of C at the specified point

$\rho = 1/\kappa$ the *radius of curvature*

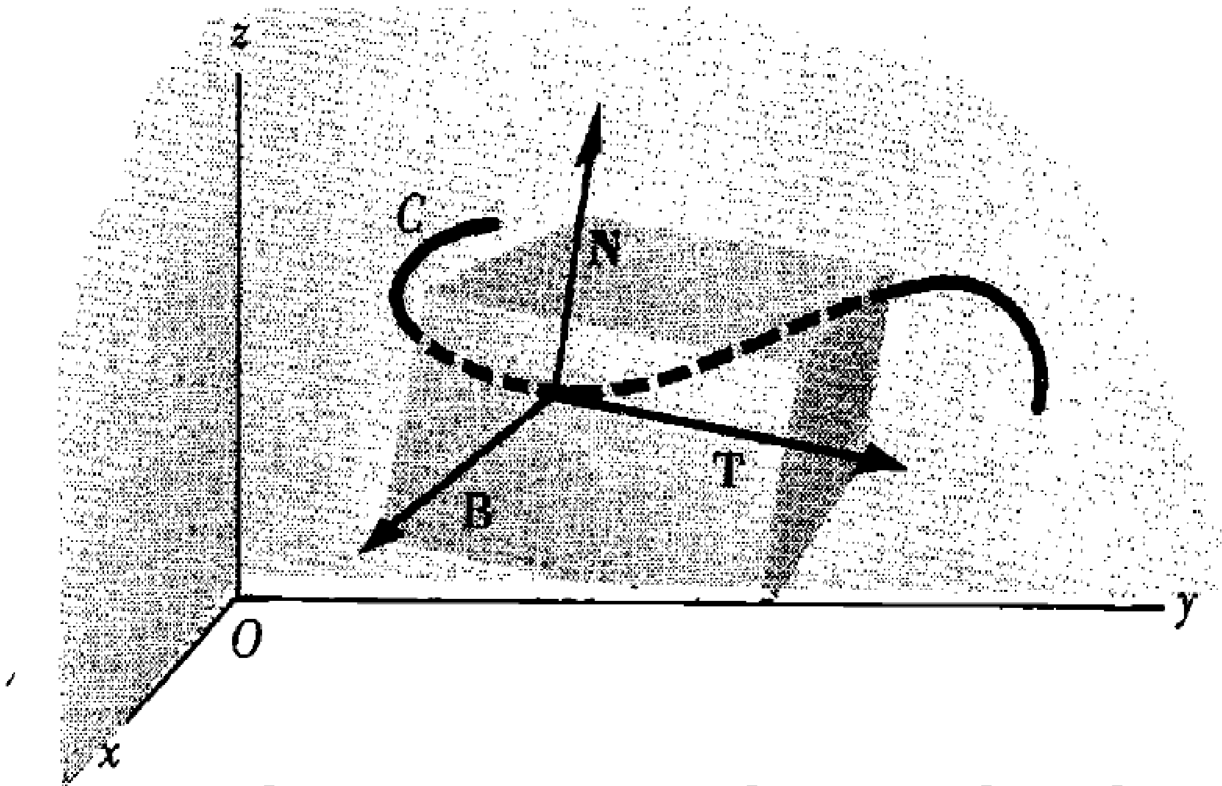
Binormal to the curve **B**

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$



T, **N**, **B** form a localized right-handed rectangular coordinate system at any specified point of C

The coordinate system {**T**, **N**, **B**} is called trihedral / triad at the point



As *s* changes, the triad {**T**, **N**, **B**} changes along the curve *C* – *moving trihedral*

Frenet-Serret formulas

$$\frac{d\mathbf{T}}{ds} = \kappa\,\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

τ is a scalar called the *torsion*

$\sigma = 1/\tau$ is called the *radius of torsion*

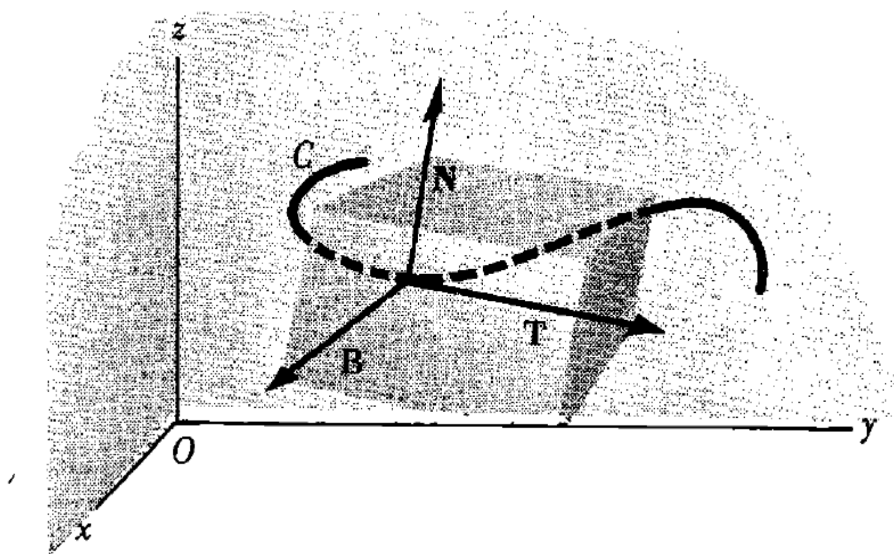
To characterise the geometry of a curve in 3D space

- Two quantities are required to describe the geometry of a 3D curve
-

τ torsion

↓

$$\frac{d\mathbf{B}}{ds}$$



κ curvature

↓

$$\frac{d\mathbf{T}}{ds}$$

It tells you how **B** varies with *s*

It tells you how **T** varies with *s*

18. Prove the Frenet-Serret formulas (a) $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$

$$\mathbf{T}\cdot\mathbf{T} = 1$$

$$\mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0$$

$\frac{d\mathbf{T}}{ds}$ is perpendicular to $\mathbf{T}$

If $\mathbf{N}$ is a unit vector in the direction $\frac{d\mathbf{T}}{ds}$, then $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$

18. Prove the Frenet-Serret formulas (*b*) $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$

To show this, need to prove the following:

$$\frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{B}$$

$$\frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{T}$$

Since {**T**, **N**, **B**} forms a right-handed system, a vector which is simultaneously perpendicular to both **B** and **T** necessarily means it is pointing in the **N** direction.

First, show $\frac{d\,\boldsymbol{B}}{ds} \perp \boldsymbol{B}$

$$\frac{d(\boldsymbol{B}\cdot\boldsymbol{B})}{ds} = \frac{d\,\boldsymbol{B}}{ds}\cdot\boldsymbol{B} + \boldsymbol{B}\cdot\frac{d\,\boldsymbol{B}}{ds} = 2\,\boldsymbol{B}\cdot\frac{d\,\boldsymbol{B}}{ds}$$

Independently,

$$\frac{d(\boldsymbol{B}\cdot\boldsymbol{B})}{ds} = \frac{d(B^2)}{ds} = 2\,B\frac{d\,B}{ds}$$

$$2\,\boldsymbol{B}\cdot\frac{d\,\boldsymbol{B}}{ds} = 2\,B\frac{d\,B}{ds} = 0$$

$$because\ \frac{d\,B}{ds} = \frac{d|\boldsymbol{B}|}{ds} = 0 \qquad \Rightarrow \frac{d\,\boldsymbol{B}}{ds} \perp \boldsymbol{B}$$

Now show $\frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{T}$

$$By\ definition,\ \boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$$

$$\frac{d\boldsymbol{B}}{ds}=\left(\frac{d\boldsymbol{T}}{ds}\times\boldsymbol{N}\right)+\left(\boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds}\right)$$

$$=\left(\kappa\,\boldsymbol{N}\times\boldsymbol{N}\right)+\left(\boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds}\right)$$

$$=\boldsymbol{T}\times\frac{d\boldsymbol{N}}{ds}$$

$$\frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{T}$$

$$\frac{d\boldsymbol{B}}{ds} = \boldsymbol{T} \times \frac{d\boldsymbol{N}}{ds}$$

$$\Rightarrow \frac{d\boldsymbol{B}}{ds} \perp \boldsymbol{T}$$

The above relation can be deduced immediately (from the properties of cross product). Can you see how?

If you want to prove it explicitly, this is how

$$\boldsymbol{T} \cdot \frac{d\boldsymbol{B}}{ds} = \boldsymbol{T} \cdot \left(\boldsymbol{T} \times \frac{d\boldsymbol{N}}{ds}\right) = \left(\boldsymbol{T} \times \boldsymbol{T}\right) \cdot \frac{d\boldsymbol{N}}{ds} = 0$$

We have shown that d**B**/d*s* is simultaneously perpendicular to both **B** and **T.** This means it is pointing in the **N** direction, since {**T**, **N**, **B**} forms a right-handed system.

$$\frac{d\boldsymbol{B}}{ds} = -\tau \boldsymbol{N}$$

We call **B** the *binormal*, τ the *torsion*

18. Prove the Frenet-Serret formulas (c) $\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}$

Don't simply prove it using brute force.

Prove it using the proof you have got in (a) and (b), namely,

$$(a)\ \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad (b)\ \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

and use the following "ingredients"

{**T**, **N**, **B**} forms a right-handed system

$$3.\quad \frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

18. Prove the Frenet-Serret formulas (c) $\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}$

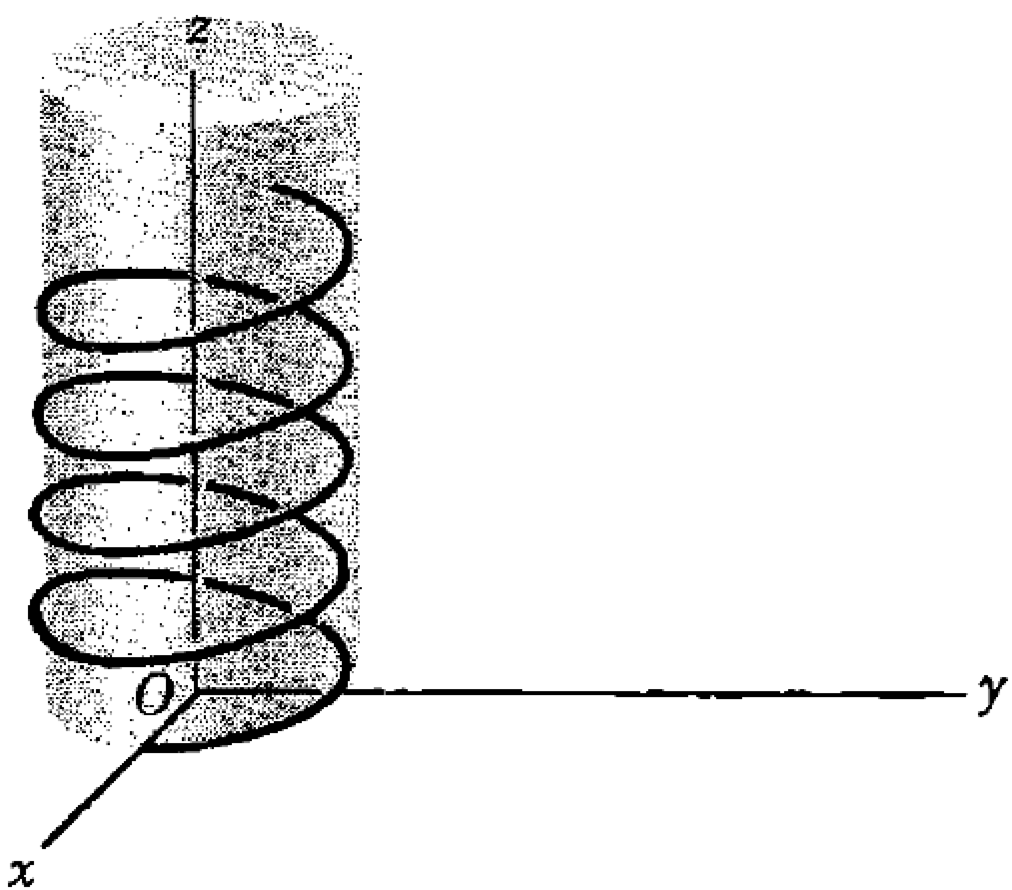
$$N = B \times T$$

$$\frac{d\mathbf{N}}{ds} = \mathbf{B} \times \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{B}}{ds} \times \mathbf{T}$$

$$= \mathbf{B} \times \kappa\mathbf{N} - \tau\mathbf{N} \times \mathbf{T}$$

$$= -\kappa\mathbf{T} + \tau\mathbf{B} = \tau\mathbf{B} - \kappa\mathbf{T}.$$

19. Sketch the space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ and find (*a*) the unit tangent **T**, (*b*) the principal normal **N**, curvature κ and radius of curvature ρ, (*c*) the binormal **B**, torsion τ and radius of torsion σ.



(a) The position vector for any point on the curve is

$$\mathbf{r} = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 4t\,\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 4\,\mathbf{k}$$

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}}$$

$$= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = 5$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt}$$

$$= -\frac{3}{5}\sin t\,\mathbf{i} + \frac{3}{5}\cos t\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$$

(b) $\frac{d\mathbf{T}}{dt} = -\frac{3}{5}\cos t\,\mathbf{i} - \frac{3}{5}\sin t\,\mathbf{j}$

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = -\frac{3}{25}\cos t\,\mathbf{i} - \frac{3}{25}\sin t\,\mathbf{j}$$

Since $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$

$$\left|\frac{d\mathbf{T}}{ds}\right| = |\kappa|\,|\mathbf{N}| = \kappa \quad \text{as } \kappa \geqq 0$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{3}{25} \quad \text{and} \quad \rho = \frac{1}{\kappa} = \frac{25}{3}$$

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

(c) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{4}{5}\sin t\,\mathbf{i} - \frac{4}{5}\cos t\,\mathbf{j} + \frac{3}{5}\mathbf{k}$$

Chapter 4

Gradient, divergence and curl

THE VECTOR DIFFERENTIAL OPERATOR DEL,

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The operator ∇ is also known as *nabla*.

This vector operator possesses properties analogous to those of ordinary vectors.

$\nabla\phi$

Let $\phi(x, y, z)$

defines a differentiable scalar field

$\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi$$

$$= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Del operator converts a scalar field into a vector field

$\phi(x, y, z)$ defines a differentiable scalar field

$\nabla\phi$ defines a vector field

Directive unit vector

The component of $\nabla\phi$ in the direction of a unit vector **a** is given by $\nabla\phi \cdot \mathbf{a}$

directional derivative of ϕ in the direction **a**.

$\nabla\phi \cdot \mathbf{a}$ is

the rate of change of ϕ at (x, y, z) in the direction **a**

For example, **a=i**

$$\nabla \phi(x,y,z) = \frac{\partial \phi(x,y,z)}{\partial x}\boldsymbol{i} + \frac{\partial \phi(x,y,z)}{\partial y}\boldsymbol{j} + \frac{\partial \phi(x,y,z)}{\partial y}\boldsymbol{k}$$

$$(\nabla \phi)\cdot \boldsymbol{i} = \frac{\partial \phi(x,y,z)}{\partial x}$$

Rate of change of ϕ at (x,y,z) along the direction **i**

THE DIVERGENCE

$$\mathbf{V}(x, y, z) = V_1\,\mathbf{i} + V_2\,\mathbf{j} + V_3\,\mathbf{k}$$

the *divergence* of **V**, written $\nabla \cdot \mathbf{V}$ or div **V**,

$$\nabla \cdot \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(V_1\,\mathbf{i} + V_2\,\mathbf{j} + V_3\,\mathbf{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$\nabla \cdot \mathbf{V} \neq \mathbf{V} \cdot \nabla$$

A scalar function ($\nabla \cdot \mathbf{V}$)

A scalar operator ($\mathbf{V} \cdot \nabla$)

THE CURL

$\mathbf{V}(x, y, z)$ a differentiable vector field

the *curl* or *rotation* of $\mathbf{V}$

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

FORMULAS INVOLVING ∇

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

2. $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$

3. $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$

4. $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$

5. $\nabla\times(\phi\mathbf{A}) = (\nabla\phi)\times\mathbf{A} + \phi(\nabla\times\mathbf{A})$

FORMULAS INVOLVING ∇

6. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

7. $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$

8. $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$

9. $\nabla\cdot(\nabla\phi) \equiv \nabla^2\phi \equiv \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

   where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the *Laplacian*

10. $\nabla\times(\nabla\phi) = \mathbf{0}$. The curl of the gradient of ϕ is zero.

11. $\nabla\cdot(\nabla\times\mathbf{A}) = 0$. The divergence of the curl of $\mathbf{A}$ is zero.

12. $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

2. Prove (a) $\nabla(F+G) = \nabla F + \nabla G$, (b) $\nabla(FG) = F\,\nabla G + G\,\nabla F$ where F and G are differentiable scalar functions of x, y and z

This is just an algebraic exercise. But still you got to make sure you know how to do it despite the straightforwardness.

2. Prove (a) $\nabla(F+G) = \nabla F + \nabla G$, (b) $\nabla(FG) = F\,\nabla G + G\,\nabla F$ where F and G are differentiable scalar functions of x, y and z

$$(a)\quad \nabla(F+G) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(F+G)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(F+G) + \mathbf{j}\frac{\partial}{\partial y}(F+G) + \mathbf{k}\frac{\partial}{\partial z}(F+G)$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)F + \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)G$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)F + \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)G$$

2. Prove (a) $\nabla(F+G) = \nabla F + \nabla G$, (b) $\nabla(FG) = F\,\nabla G + G\,\nabla F$ where F and G are differentiable scalar functions of x, y and z

$$(b)\ \nabla(FG) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(FG)$$

$$= \frac{\partial}{\partial x}(FG)\mathbf{i} + \frac{\partial}{\partial y}(FG)\mathbf{j} + \frac{\partial}{\partial z}(FG)\mathbf{k}$$

$$= \left(F\frac{\partial G}{\partial x} + G\frac{\partial F}{\partial x}\right)\mathbf{i} + \left(F\frac{\partial G}{\partial y} + G\frac{\partial F}{\partial y}\right)\mathbf{j} + \left(F\frac{\partial G}{\partial z} + G\frac{\partial F}{\partial z}\right)\mathbf{k}$$

$$= F\left(\frac{\partial G}{\partial x}\mathbf{i} + \frac{\partial G}{\partial y}\mathbf{j} + \frac{\partial G}{\partial z}\mathbf{k}\right) + G\left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right)$$

$$= F\nabla G + G\nabla F$$

3. Find $\nabla\phi$ if (*a*) $\phi = \ln|\mathbf{r}|$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \ln|\mathbf{r}| = \tfrac{1}{2}\ln(x^2 + y^2 + z^2)$$

$$\nabla\phi = \tfrac{1}{2}\nabla\ln(x^2 + y^2 + z^2)$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right\}$$

$$= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

3. Find $\nabla\phi$ if (b) $\phi = \frac{1}{r}$.

$$\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \nabla\left\{(x^2+y^2+z^2)^{-1/2}\right\}$$

$$= \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \mathbf{i}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + \mathbf{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\} + \mathbf{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\}$$

$$= \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

Can you recognize these?

$$\phi = \frac{1}{r}$$

$$\nabla\phi = -\frac{\mathbf{r}}{r^3}$$

Of course, you have already seen them before (apart from a constant) in at least two instances

Can you recognize these?

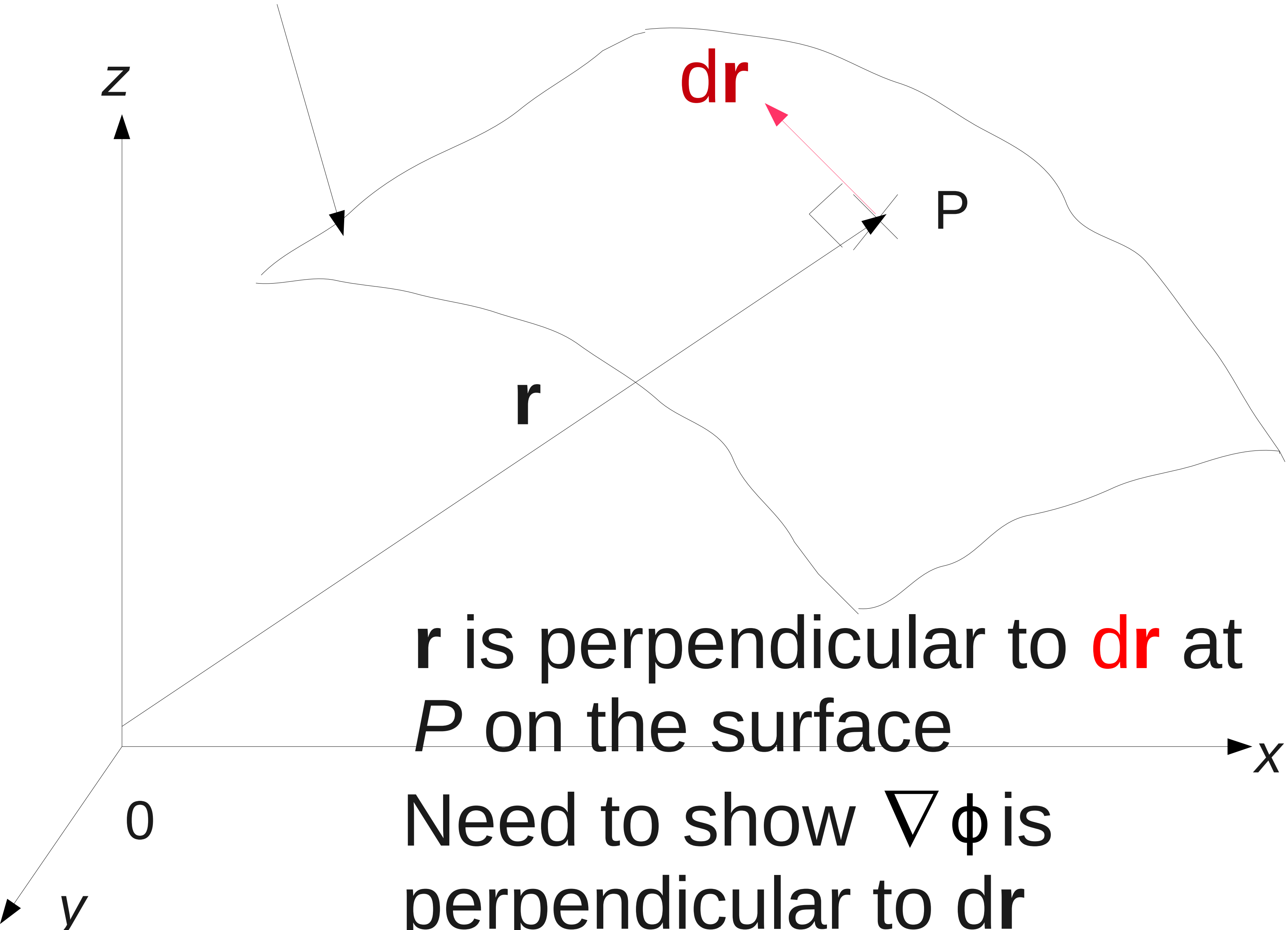
$$\phi = -\frac{1}{r}$$

$$-\nabla\phi = \frac{\mathbf{r}}{r^3} = \frac{\hat{\boldsymbol{r}}}{r^2}$$

$\varphi(\mathbf{r})$: spherically symmetric gravitational/Coulomb potential

$-\nabla\phi$: gravitational field / Coulomb field

5. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.



r is perpendicular to **dr** at *P* on the surface

Need to show $\nabla\phi$ is perpendicular to d**r**

Show $\nabla\phi$ is perpendicular to d**r**

$$\nabla\phi \cdot d\mathbf{r} =$$

$$\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot\left(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}\right)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi$$

$$= 0 \quad \text{since} \quad \phi(x,y,z) = c$$

$\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point (2,—2,3).

Use the previous result. $\nabla\phi \cdot d\mathbf{r} = 0$

$\nabla\phi$ is perpendicular to the surface

Find $\nabla\phi$ at P(2,-2,3). It is a vector normal to the surface at *P*.

The normal unit vector at *P* can then be obtained via $\nabla\phi \big/ |\nabla\phi|$

$$\nabla(x^2y + 2xz)$$

$$= -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \quad \text{at the point } (2,-2,3)$$

a unit normal to the surface =

$$\frac{-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{(-2)^2 + (4)^2 + (4)^2}} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

Is there any other unit normal?

12. Find the angle between the surfaces

$$x^2+y^2+z^2=9 \text{ and } z=x^2+y^2-3$$

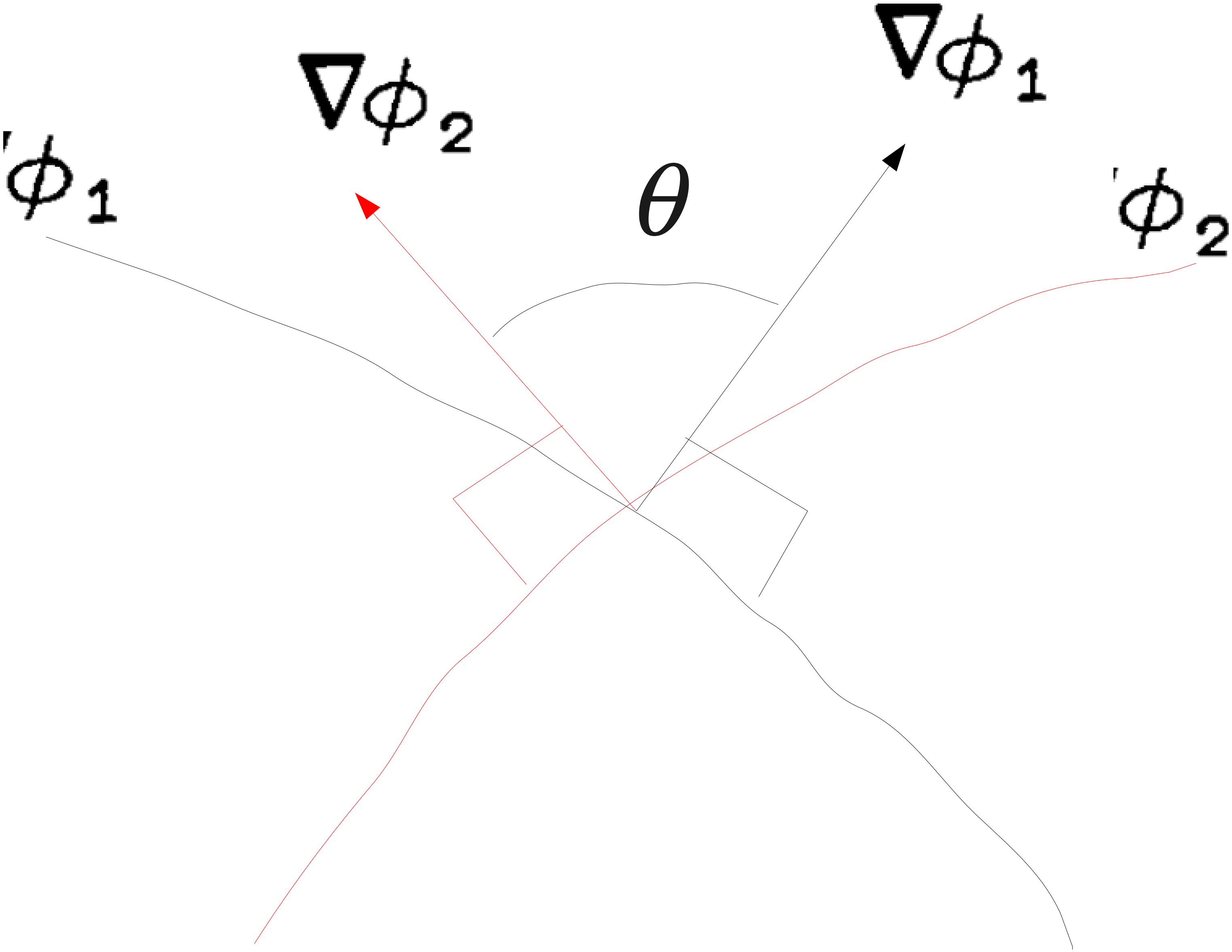
at the point $(2,-1,2)$.

See mathematica file chap4.nb for the looks of these surfaces

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

$$\nabla\phi_2 \qquad \nabla\phi_1$$

Simply: find the angle between these two vectors


$\nabla\phi_2$
$\nabla\phi_1$
ϕ_1
θ
ϕ_2

at $(2,-1,2)$

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta$$

$$\cos\theta = 0.5819$$

$$\theta = \text{arc}\cos 0.5819 = 54^\circ 25'$$

See mathematica code, Ch4.nb, to visualise these surfaces and the the normals

15. If $\mathbf{A} = x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}$, find $\nabla\cdot\mathbf{A}$ (or div $\mathbf{A}$) at the point $(1,-1,1)$.

This is a simply a simple algebraic example to show how divergence work on a vector field.

$$\nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2 = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3$$

16. (*b*) Show that $\nabla \cdot \nabla \phi = \nabla^2 \phi$,

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla \cdot \nabla \phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$$

17. Prove that $\nabla^2(\frac{1}{r}) = 0.$

$$\nabla^2(\frac{1}{r}) = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})(\frac{1}{\sqrt{x^2+y^2+z^2}})$$

$$\frac{\partial}{\partial x}(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \frac{\partial}{\partial x}\left[-x\,(x^2+y^2+z^2)^{-3/2}\right]$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2 - z^2 - x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$$

$\nabla^2\phi = 0$ is called *Laplace's equation.*

$\phi = 1/r$ is a solution of this equation.

Gauss's law in integral form (Electrostatics)

$$\int_{Gaussian\,surface} \boldsymbol{E} \cdot d\,\boldsymbol{A} = \frac{\rho}{\epsilon_0}$$

Gauss law for electrostatic field

For Gaussian surface enclosing zero net charge, Gauss law in the integral form reduce to

$$\int_{Gaussian\,surface} \boldsymbol{E} \cdot d\,\boldsymbol{A} = 0$$

Gauss's law in differential form

$$\int_{Gaussian\,surface} \boldsymbol{E}\cdot d\,\boldsymbol{A}=0$$

You will learn in later chapter that Gauss law can be cast into an equivalent form

$$\int_{Gaussian\,surface} \boldsymbol{E}\cdot d\,\boldsymbol{A}=0 \longrightarrow \nabla\cdot\boldsymbol{E}=0$$

On the other hand, electric field **E** and the electric potential φ are related via

$$\boldsymbol{E}=-\nabla\,\phi$$

Derivation of Laplace equation

$$\nabla^2 \phi = 0$$

- Taking the divergence of the electric field, we can express the Gauss law in differential form in terms of electric potential

$$\boldsymbol{E} = -\nabla \phi$$

↓ Taking divergence

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot (-\nabla \phi)$$

$$\nabla \cdot \boldsymbol{E} = -\nabla^2 \phi = 0$$

$$because\ \nabla \cdot \boldsymbol{E} = 0$$

Solution to the Laplace equation

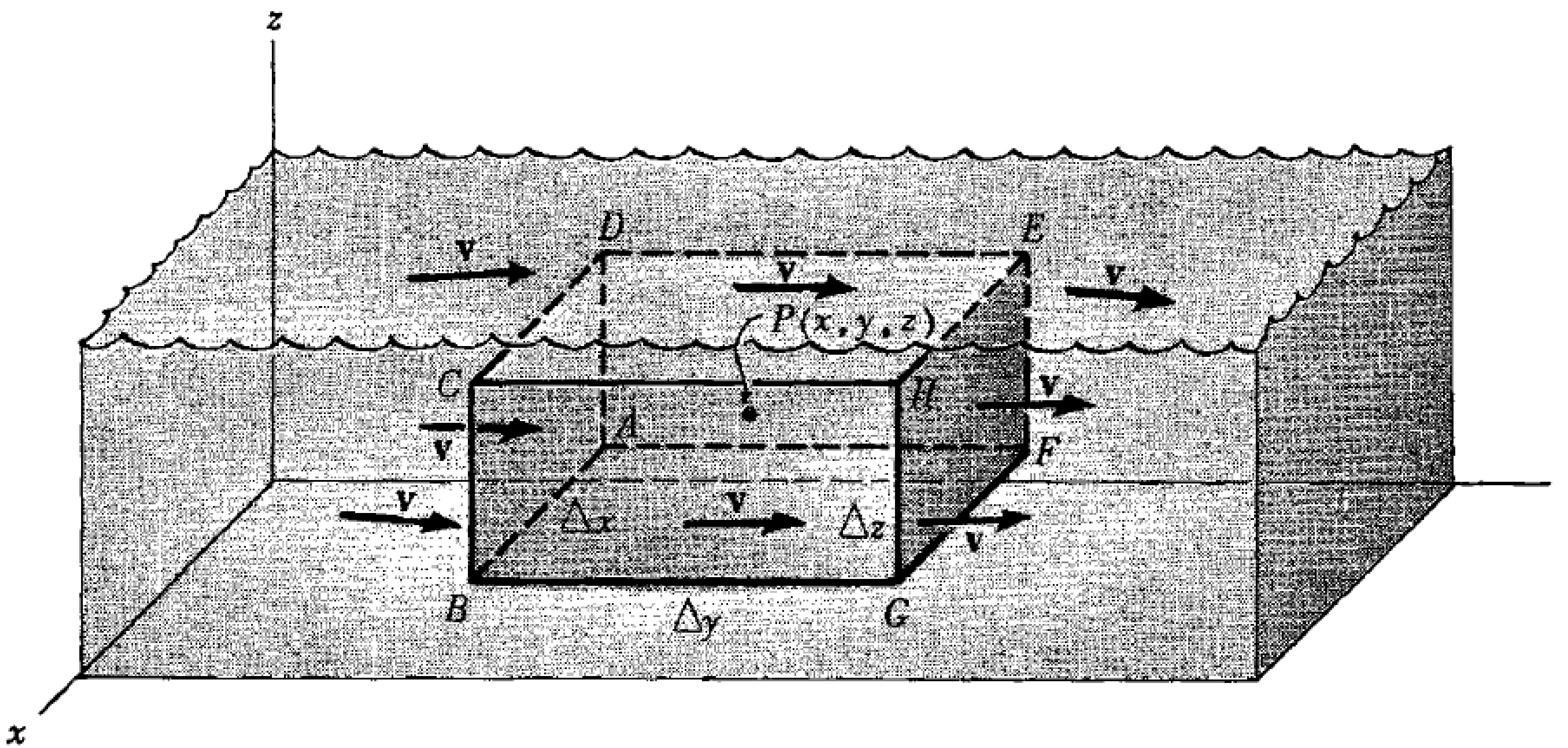
$\nabla^2 \phi = 0$ is called *Laplace's equation.*

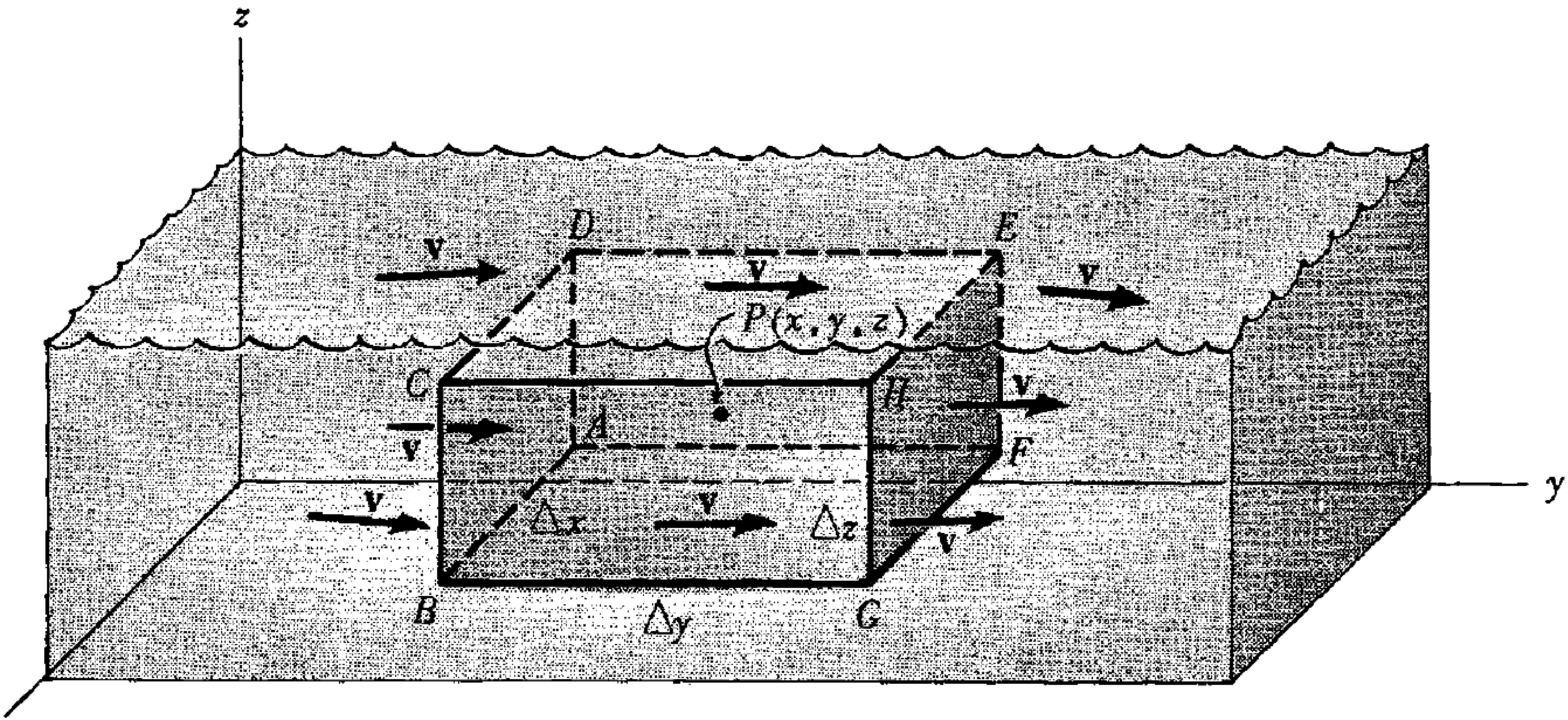
Laplace equation is just Gauss law stated in the form of the Laplacian of the electric potential. So, instead of dealing with electric field on the Gaussian surface, you just simply need to solve the Laplace equation for the potential field.

Example: Application of divergence in fluid dynamics

The flow of fluid is characterised by the velocity vector field, $\mathbf{v}(x,y,z)$

Calculate the loss in the volume element per unit time

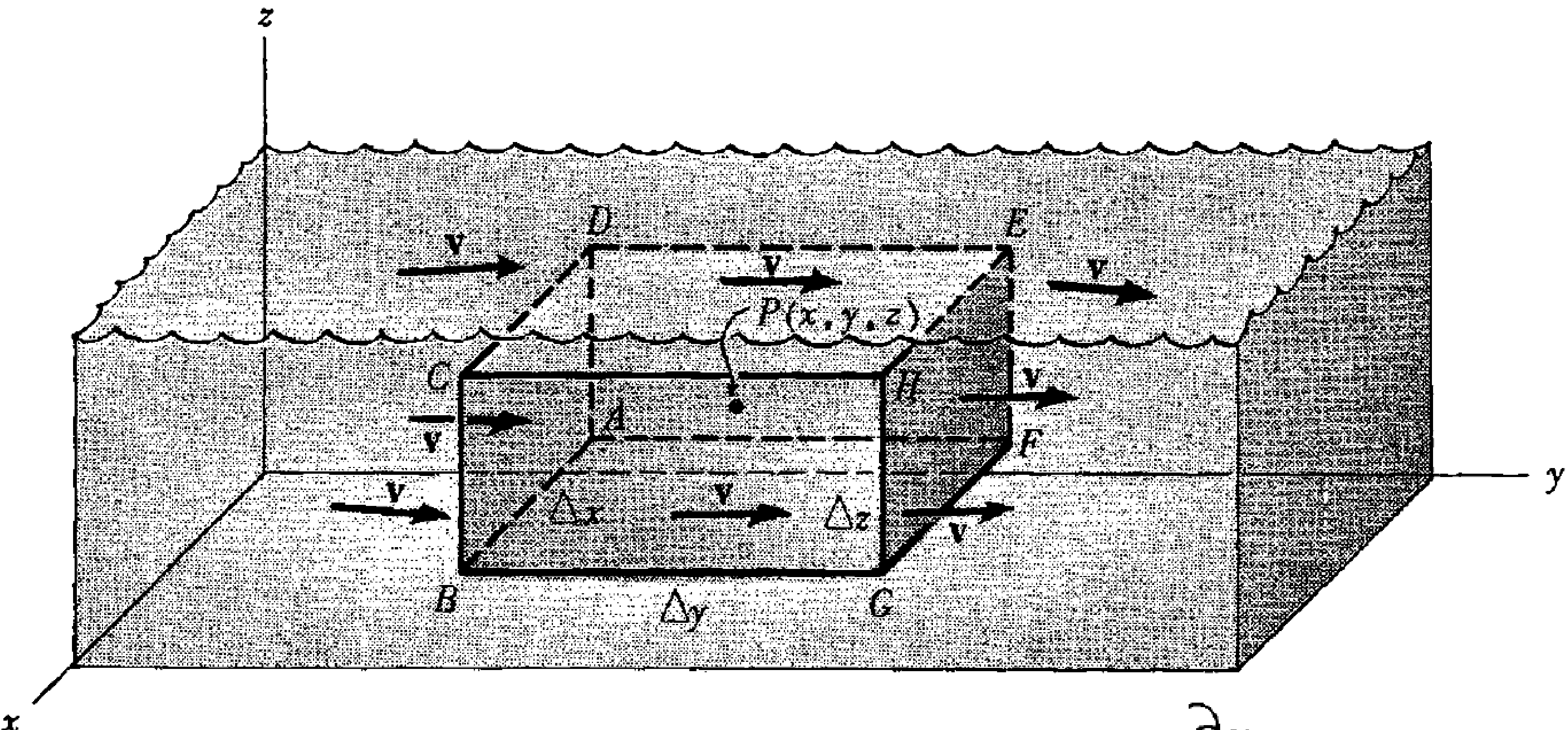




x component of velocity $\mathbf{v}$ at P $= v_1$

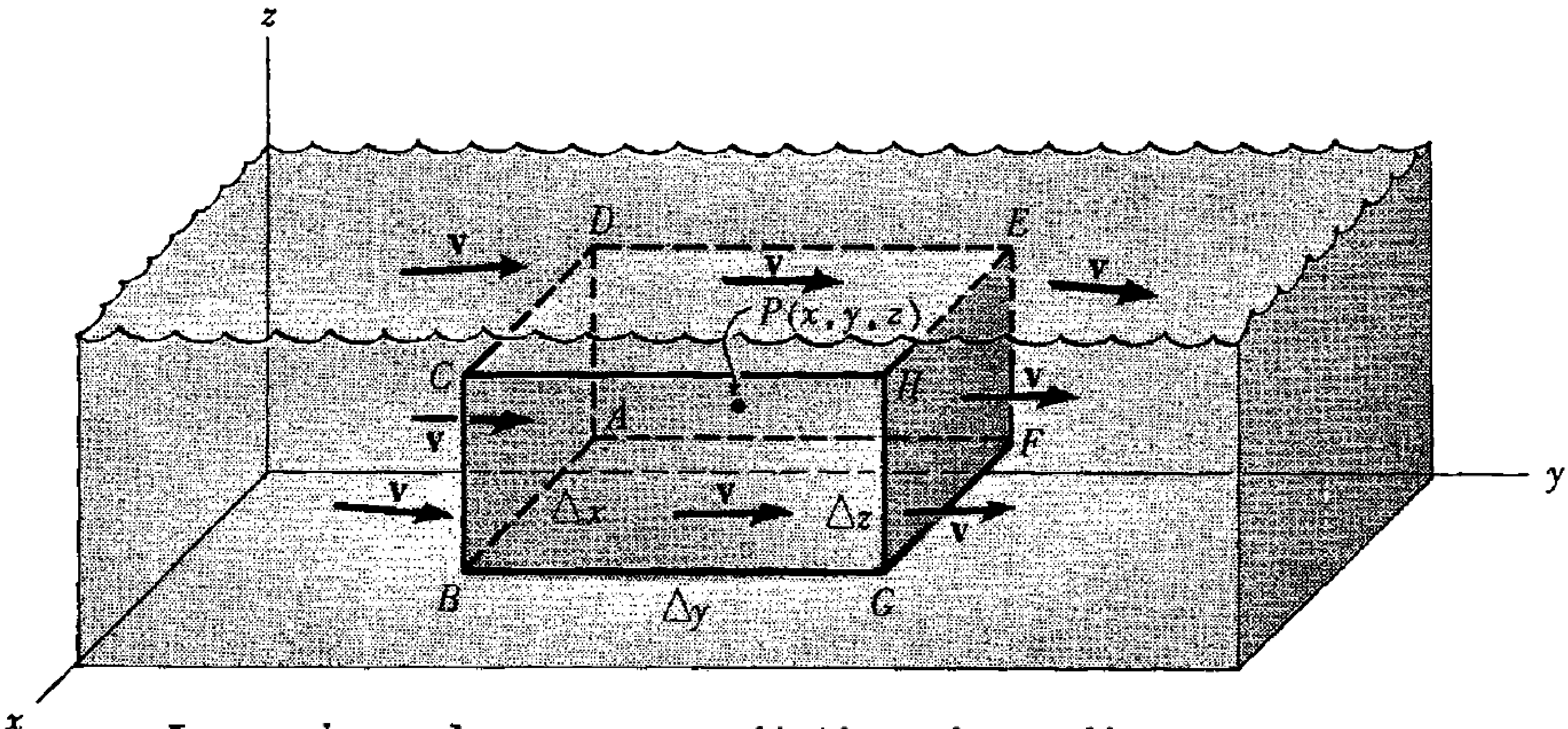
x component of $\mathbf{v}$ at center of face $AFED$ $= v_1 - \dfrac{1}{2}\dfrac{\partial v_1}{\partial x}\,\Delta x$

x component of $\mathbf{v}$ at center of face $GHCB$ $= v_1 + \dfrac{1}{2}\dfrac{\partial v_1}{\partial x}\,\Delta x$



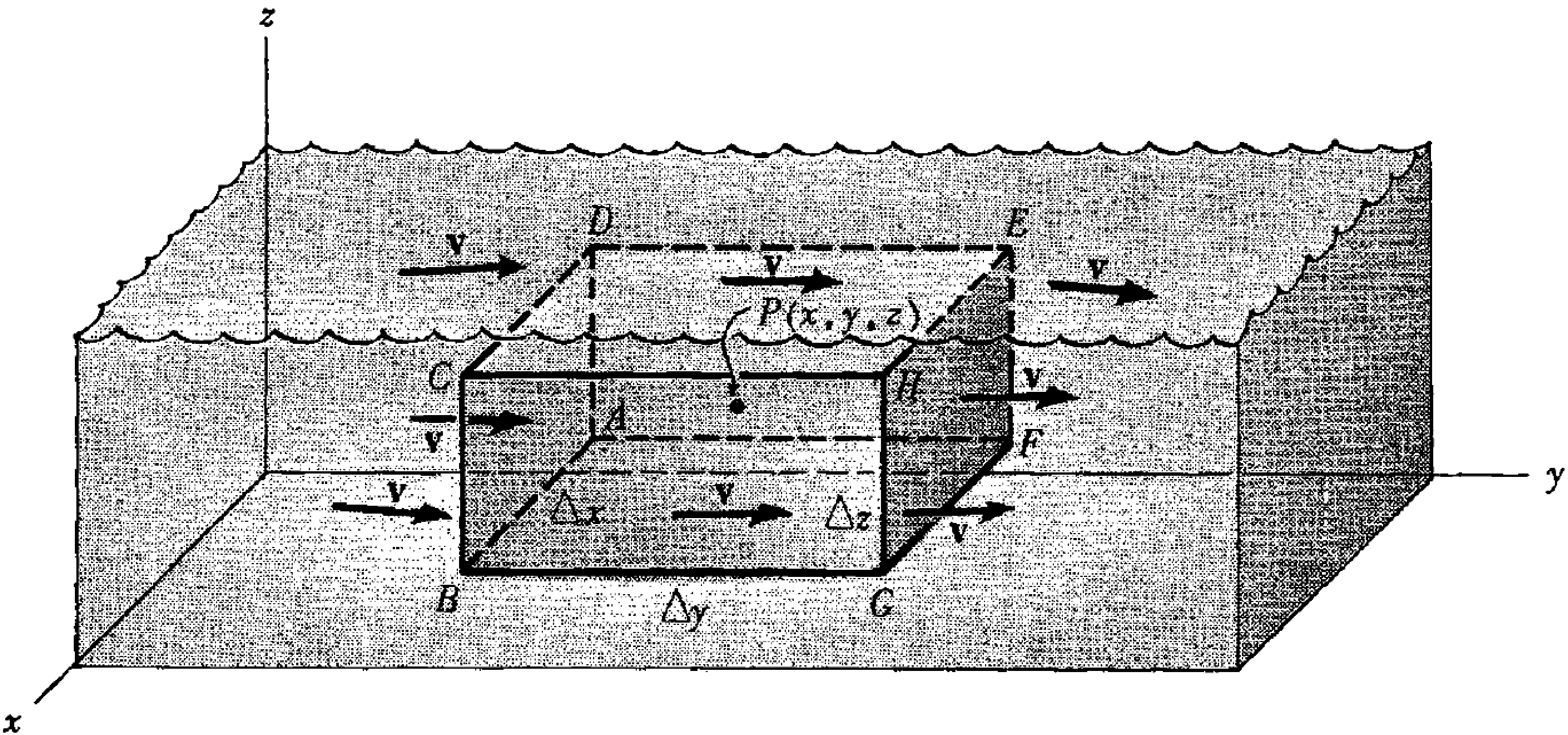
(1) volume of fluid crossing $AFED$ per unit time $\quad \left(v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y\,\Delta z$

(2) volume of fluid crossing $GHCB$ per unit time $\quad \left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y\,\Delta z$



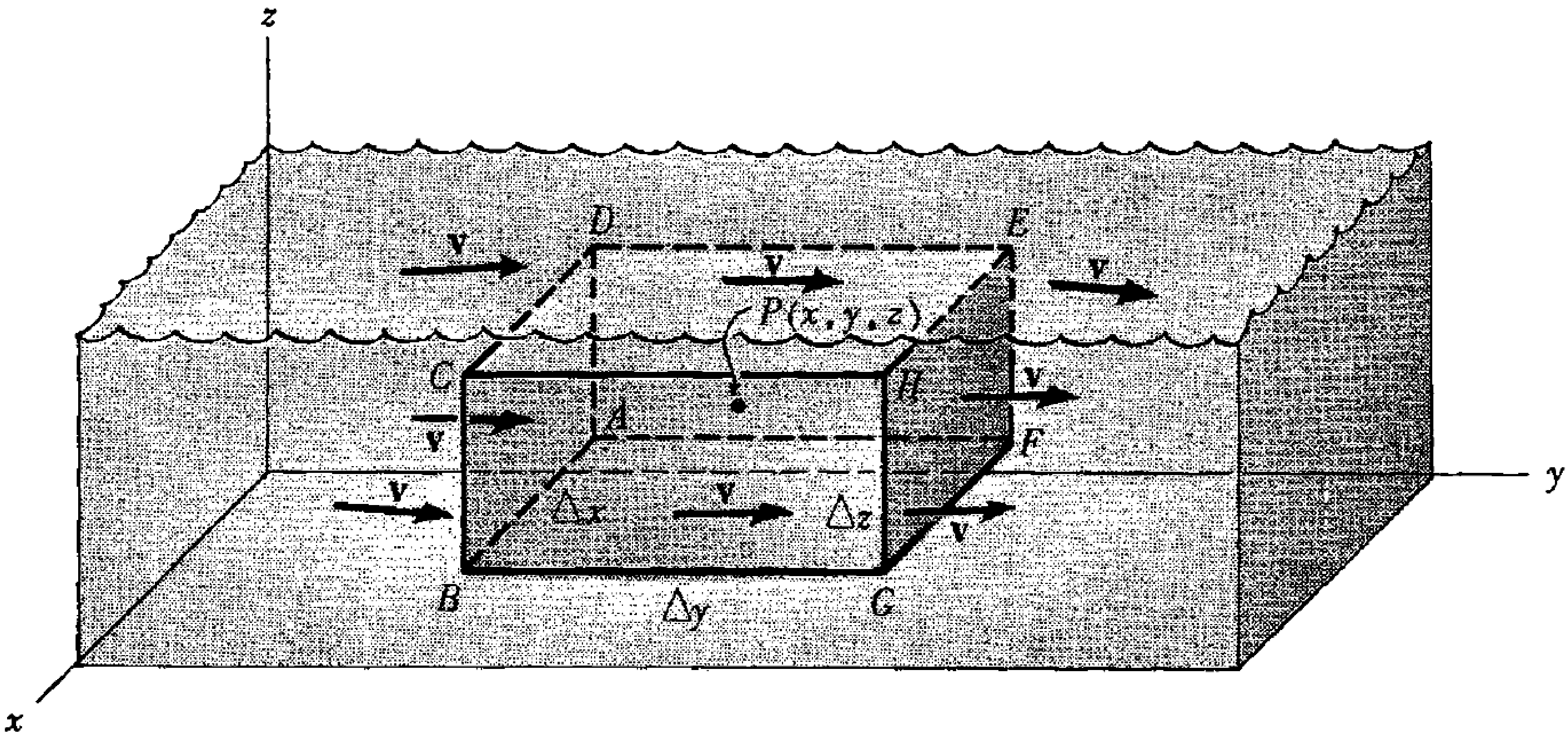
Loss in volume per unit time in x direction

$$= \quad (2) - (1) \quad = \quad \frac{\partial v_1}{\partial x} \Delta x \Delta y \Delta z$$



$$\text{loss in volume per unit time in } y \text{ direction} = \frac{\partial v_2}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\text{loss in volume per unit time in } z \text{ direction} = \frac{\partial v_3}{\partial z} \Delta x \, \Delta y \, \Delta z$$



total loss in volume per unit volume per unit time

$$\frac{\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z} = \text{div}\ \mathbf{v} = \nabla\cdot\mathbf{v}$$

Continuity equation for incompressible fluid

- Incompressible fluid: fluid that is neither created nor destroyed at any point.

Hence no 'volume loss' will occur to an incompressible fluid.

$$\nabla \cdot \mathbf{v} = 0$$

Solenoidal vector

$$\nabla \cdot \mathbf{v} = 0$$

A vector **V** is solenoidal if its divergence is zero

24. If $\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find curl curl $\mathbf{A}$.

$$\text{curl curl } \mathbf{A} = \nabla \times (\nabla \times \mathbf{A})$$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \nabla \times \left[(2x + 2z)\mathbf{i} - (x^2 + 2z)\mathbf{k} \right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= (2x + 2)\mathbf{j}$$

$$\mathbf{A} = x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$$

$$\nabla \times (\nabla \times \mathbf{A}) = (2x + 2)\mathbf{j}$$

26. Evaluate $\nabla\cdot(\mathbf{A}\times\mathbf{r})$ if $\nabla\times\mathbf{A}=0$

Hint: 1. Cast $\nabla\cdot(\mathbf{A}\times\mathbf{r})$ into a form that explicitly contains $\nabla\times\mathbf{A}$

2. Use: $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{r})=\boldsymbol{r}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{r})$$

Note:

A is a general vector field;

r is position vector.

The curl of a position vector is zero

$$\nabla \times \boldsymbol{r} = 0$$

$$\nabla \times \boldsymbol{r} = \boldsymbol{i}\left[\left(\frac{\partial}{\partial y}\right) z - \left(\frac{\partial}{\partial z}\right) y\right] + $$

$$-\boldsymbol{j}\left[\left(\frac{\partial}{\partial x}\right) z - \left(\frac{\partial}{\partial z}\right) x\right] + $$

$$\boldsymbol{k}\left[\left(\frac{\partial}{\partial x}\right) y - \left(\frac{\partial}{\partial y}\right) x\right] = 0$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{r})=\boldsymbol{r}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{r})$$
$$=\boldsymbol{r}\cdot(\nabla\times\boldsymbol{A})$$

If $\nabla\times\mathbf{A}=0$

$\nabla\cdot(\mathbf{A}\times\mathbf{r})$ reduces to zero.

27. Prove: (*a*) $\nabla \times (\nabla \phi) = \mathbf{0}$ (curl grad $\phi = \mathbf{0}$)

Note: ϕ a general scalar field

$$\nabla \times (\nabla \phi) = \nabla \times \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]\mathbf{k}$$

$$= \left(\frac{\partial^2 \phi}{\partial y\, \partial z} - \frac{\partial^2 \phi}{\partial z\, \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 \phi}{\partial z\, \partial x} - \frac{\partial^2 \phi}{\partial x\, \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 \phi}{\partial x\, \partial y} - \frac{\partial^2 \phi}{\partial y\, \partial x}\right)\mathbf{k} = \mathbf{0}$$

27. Prove:

(*b*) $\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (\text{div curl } \mathbf{A} = \mathbf{0})$

Note: **A** a general vector field

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k} \right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= \frac{\partial^2 A_3}{\partial x\, \partial y} - \frac{\partial^2 A_2}{\partial x\, \partial z} + \frac{\partial^2 A_1}{\partial y\, \partial z} - \frac{\partial^2 A_3}{\partial y\, \partial x} + \frac{\partial^2 A_2}{\partial z\, \partial x} - \frac{\partial^2 A_1}{\partial z\, \partial y} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Where will you see this equation?

When you learn magnetostatics.

Magnetic field **B** is express in terms of vector potential via

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

The fact that there is no magnetic monopole is represented by the statement $\nabla \cdot \boldsymbol{B} = 0$

Hence the equation $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ follows.

30. If $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, prove $\boldsymbol{\omega} = \frac{1}{2} \operatorname{curl} \mathbf{v}$ where $\boldsymbol{\omega}$ is a constant vector.

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}\right]$$

$$= \nabla \times \left[(\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= 2(\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}) = 2\boldsymbol{\omega}$$

Then $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$.

The curl of a vector field **v**

- This problem indicates that the curl of a vector field has something to do with rotational properties of the field.
- If the field **v** is that due to a moving fluid, for example, then a paddle wheel placed at various points in the field would tend to rotate in regions where curl **v** ≠ 0 (vortex field)
- If curl **v** = 0 in the region there would be no rotation and the field **v** is then called irrotational.

The curl of a vector field **v**

- curl **v** ≠ 0: **v** is a vortex field (it 'rotates')
- curl **v** = 0: **v** is an irrotational field (it does not 'rotate')

29. Prove $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$

THE VECTOR DIFFERENTIAL OPERATOR DEL

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

possesses properties analogous to those of ordinary vectors.

So that you can prove 29 using the vector triple cross product result

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

29. Prove $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Placing $\mathbf{A} = \mathbf{B} = \nabla$ and $\mathbf{C} = \mathbf{F}$,

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

Where would you encounter the above identity?

When you learn Maxwell equations in electrodynamics.

Maxwell's equations of electromagnetic theory.

31. If $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{H}}{\partial t}$,

$\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{E}}{\partial t}$, show that **E** and **H** satisfy

$$\nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

You start from the identity proven in the previous slide and apply it on the **E** field:

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

$$\downarrow \quad \mathbf{A} \rightarrow \mathbf{E}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -\nabla^2 \mathbf{E}$$

Eq. (1)

because $\nabla \cdot \mathbf{E} = 0$

Independently,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{H}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Eq. (2)

Combining Eq. (1) and Eq. (2),

$$\text{Then } \nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly,

$$\nabla \times (\nabla \times \mathbf{H}) = -\nabla^2 \mathbf{H} + \nabla(\nabla \cdot \mathbf{H}) = -\nabla^2 \mathbf{H}$$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

$$= \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

Then $\nabla^2 \mathbf{H} = \dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$ is called the *wave equation.*

The electric and magnetic fields, **E** and **H**, propagates according to the wave equation – prediction of electromagnetic wave propagating in free space.

In other words, light is just electromagnetic wave.

Chapter 5

Vector integration

ORDINARY INTEGRALS OF VECTORS.

Let $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ be a vector depending on a single scalar variable u, where $R_1(u)$, $R_2(u)$, $R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

is called an *indefinite integral* of $\mathbf{R}(u)$.

If there exists a vector $\mathbf{S}(u)$ such that $\mathbf{R}(u) = \frac{d}{du}\left(\mathbf{S}(u)\right)$, then

$$\int \mathbf{R}(u)\,du = \int \frac{d}{du}\left(\mathbf{S}(u)\right)du = \mathbf{S}(u) + \mathbf{c}$$

where $\mathbf{c}$ is an *arbitrary constant vector* independent of u.

definite integral

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}\big(\mathbf{S}(u)\big)\,du$$

$$= \mathbf{S}(u) + \mathbf{c}\,\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

1. If $\mathbf{R}(u) = (u-u^2)\,\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$,

find (a) $\int \mathbf{R}(u)\,du$ and (b) $\int_1^2 \mathbf{R}(u)\,du$.

$$(a)\ \int \mathbf{R}(u)\,du = \int \left[(u-u^2)\,\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}\right] du$$

$$= \mathbf{i}\int (u-u^2)\,du + \mathbf{j}\int 2u^3\,du + \mathbf{k}\int -3\,du$$

$$= \mathbf{i}\left(\frac{u^2}{2} - \frac{u^3}{3} + c_1\right) + \mathbf{j}\left(\frac{u^4}{2} + c_2\right) + \mathbf{k}\left(-3u + c_3\right)$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\,\mathbf{j} - 3u\,\mathbf{k} + c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\,\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}$$

$$\int_1^2 \mathbf{R}(u)\,du = \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}\Big|_1^2$$

$$= \left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right)\mathbf{i} + \frac{2^4}{2}\mathbf{j} - 3(2)\mathbf{k} + \mathbf{c}\right] -$$

$$\left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right)\mathbf{i} + \frac{1^4}{2}\mathbf{j} - 3(1)\mathbf{k} + \mathbf{c}\right]$$

$$= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

Another Method.

$$\int_1^2 \mathbf{R}(u)\,du = \mathbf{i}\int_1^2 (u-u^2)\,du + \mathbf{j}\int_1^2 2u^3\,du + \mathbf{k}\int_1^2 -3\,du$$

$$= \mathbf{i}\left(\frac{u^2}{2}-\frac{u^3}{3}\right)\Big|_1^2 + \mathbf{j}\left(\frac{u^4}{2}\right)\Big|_1^2 + \mathbf{k}(-3u)\Big|_1^2$$

$$= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

2. The acceleration of a particle at any time $t \geqq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

Integrating,

$$\mathbf{v} = \mathbf{i}\int 12\cos 2t\,dt + \mathbf{j}\int -8\sin 2t\,dt + \mathbf{k}\int 16t\,dt$$

$$= 6\sin 2t\,\mathbf{i} + 4\cos 2t\,\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}_1$$

Putting $\mathbf{v}=\mathbf{0}$ when $t=0$,

we find $\mathbf{0} = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c}_1$ and $\mathbf{c}_1 = -4\mathbf{j}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k}.$$

Integrating,

$$\mathbf{r} = \mathbf{i}\int 6\sin 2t\,dt + \mathbf{j}\int (4\cos 2t - 4)\,dt + \mathbf{k}\int 8t^2\,dt$$

$$= -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + \mathbf{c}_2$$

Putting $\mathbf{r} = \mathbf{0}$ when $t = 0$, $\mathbf{c}_2 = 3\,\mathbf{i}$

$$\mathbf{r} = (3 - 3\cos 2t)\mathbf{i} + (2\sin 2t - 4t)\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k}$$

3. Evaluate $\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt$.

The trick is: do not integrate directly by brute force. Instead, cast the integrant into the form of a differentiation

$$\int [\boldsymbol{A} \times \frac{d^2 \boldsymbol{A}}{dt^2}] dt = \int [\frac{d}{dt}(something)] dt$$

so that

$$\int [\frac{d}{dt}(something)] dt = something + constant\ vector$$

3. Evaluate $\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt$.

$$\frac{d}{dt}(\mathbf{A} \times \frac{d\mathbf{A}}{dt}) = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2} + \frac{d\mathbf{A}}{dt} \times \frac{d\mathbf{A}}{dt} = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}$$

$$\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt = \int \frac{d}{dt}(\mathbf{A} \times \frac{d\mathbf{A}}{dt})\, dt$$

$$= \mathbf{A} \times \frac{d\mathbf{A}}{dt} + \mathbf{c} .$$

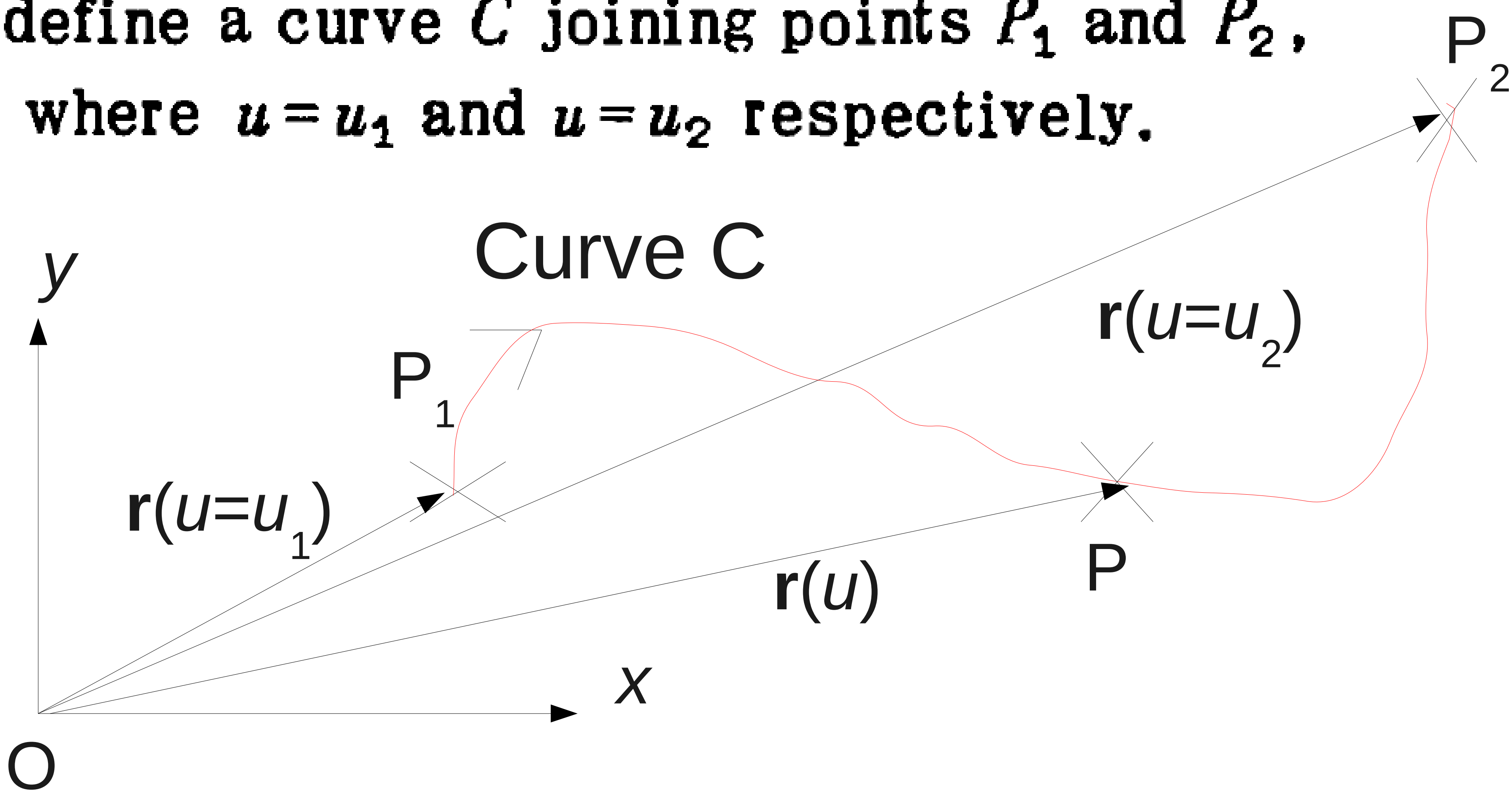
LINE INTEGRALS.

Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$,

where $\mathbf{r}(u)$ is the position vector of (x, y, z),

define a curve C joining points P_1 and P_2,

where $u = u_1$ and $u = u_2$ respectively.



Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$
be a vector function of position defined and continuous along C.

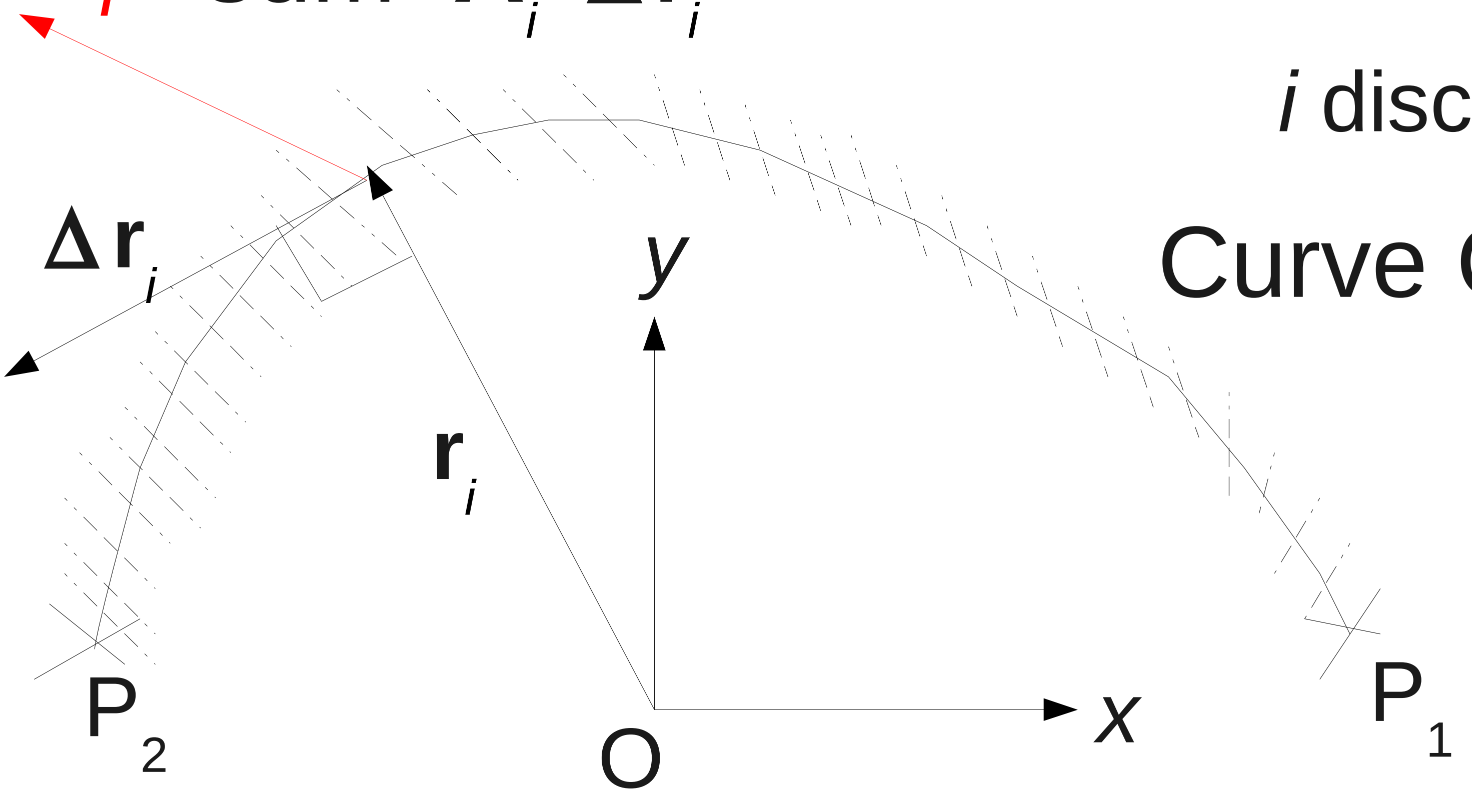
Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r}$$

$$= \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral.*

Line integration as the limit of discrete sum $\mathbf{A}_i \bullet \Delta \mathbf{r}_i$



i discrete variable

$\Delta \mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$

$$\lim_{N \to \infty} \sum_{i=1}^{i=N} \boldsymbol{A}_i \cdot \Delta \boldsymbol{r}_i = \int_{P_1}^{P_2} \boldsymbol{A} \cdot d\boldsymbol{r} = \int_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

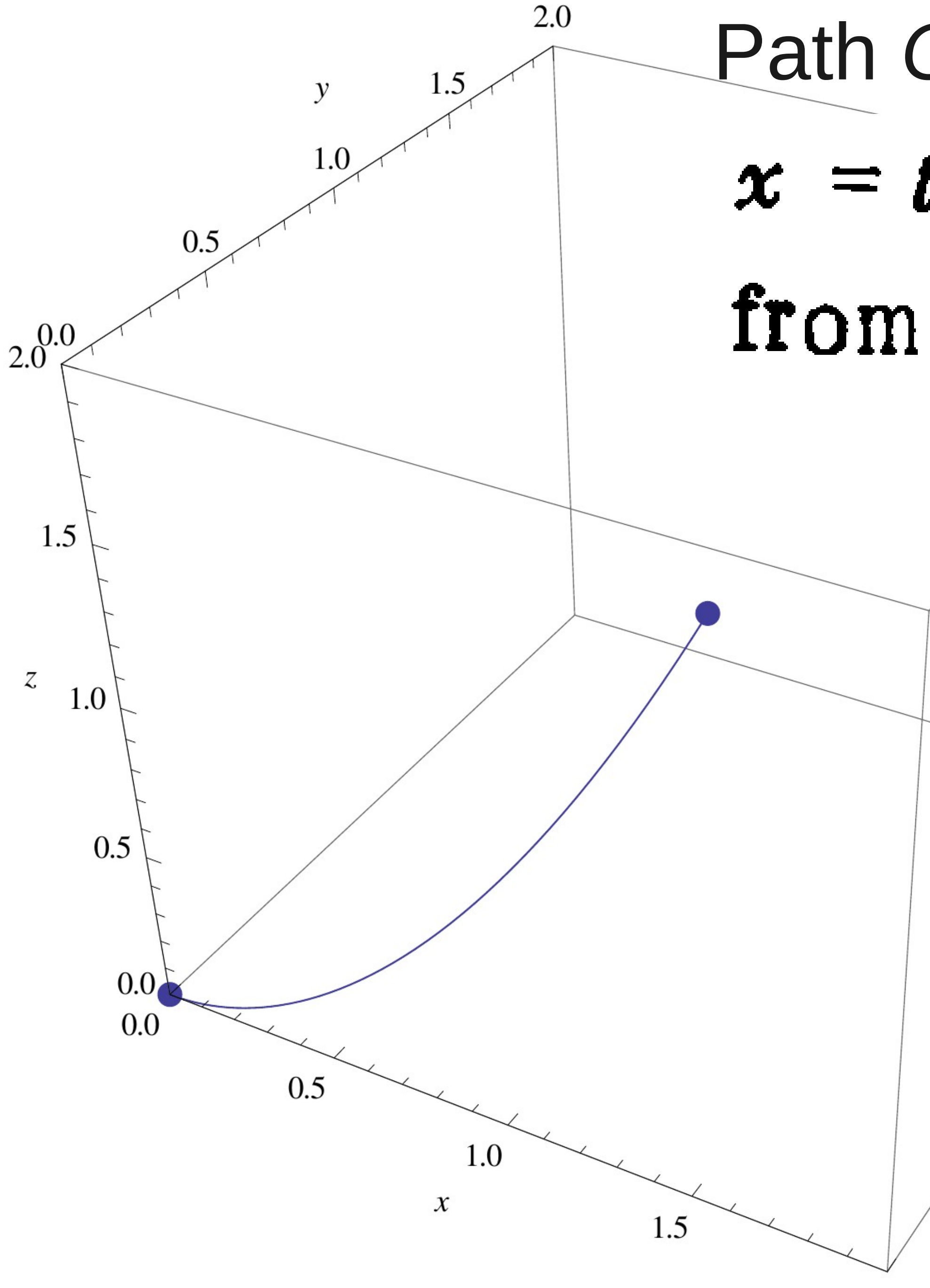
6. If $\mathbf{A} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$.

Path C, as defined by

$$x = t, \; y = t^2, \; z = t^3$$

from (0,0,0) to (1,1,1)



When the parameter t varies from 0 to 1, a point P traverses from (0,0,0) to (1,1,1) along the curve C

See mathematica code, Ch5.nb

$$\int_C \mathbf{A} \cdot d\mathbf{r} =$$

$$\int_C \left[(3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}\right] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

If $x = t,\ y = t^2,\ z = t^3$,

points (0,0,0) and (1,1,1) correspond to $t = 0$ and $t = 1$ respectively.

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2 + 6t^2)\, dt - 14(t^2)(t^3)\, d(t^2) + 20(t)(t^3)^2\, d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\, dt - 28t^6\, dt + 60t^9\, dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\, dt$$

$$= 3t^3 - 4t^7 + 6t^{10} \Big|_0^1 = 5$$

Another Method.

$$\mathbf{A} = (3x^2+6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$$

Along C, $x = t$, $y = t^2$, $z = t^3$

$$\mathbf{A} = 9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k}$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

$$d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$$

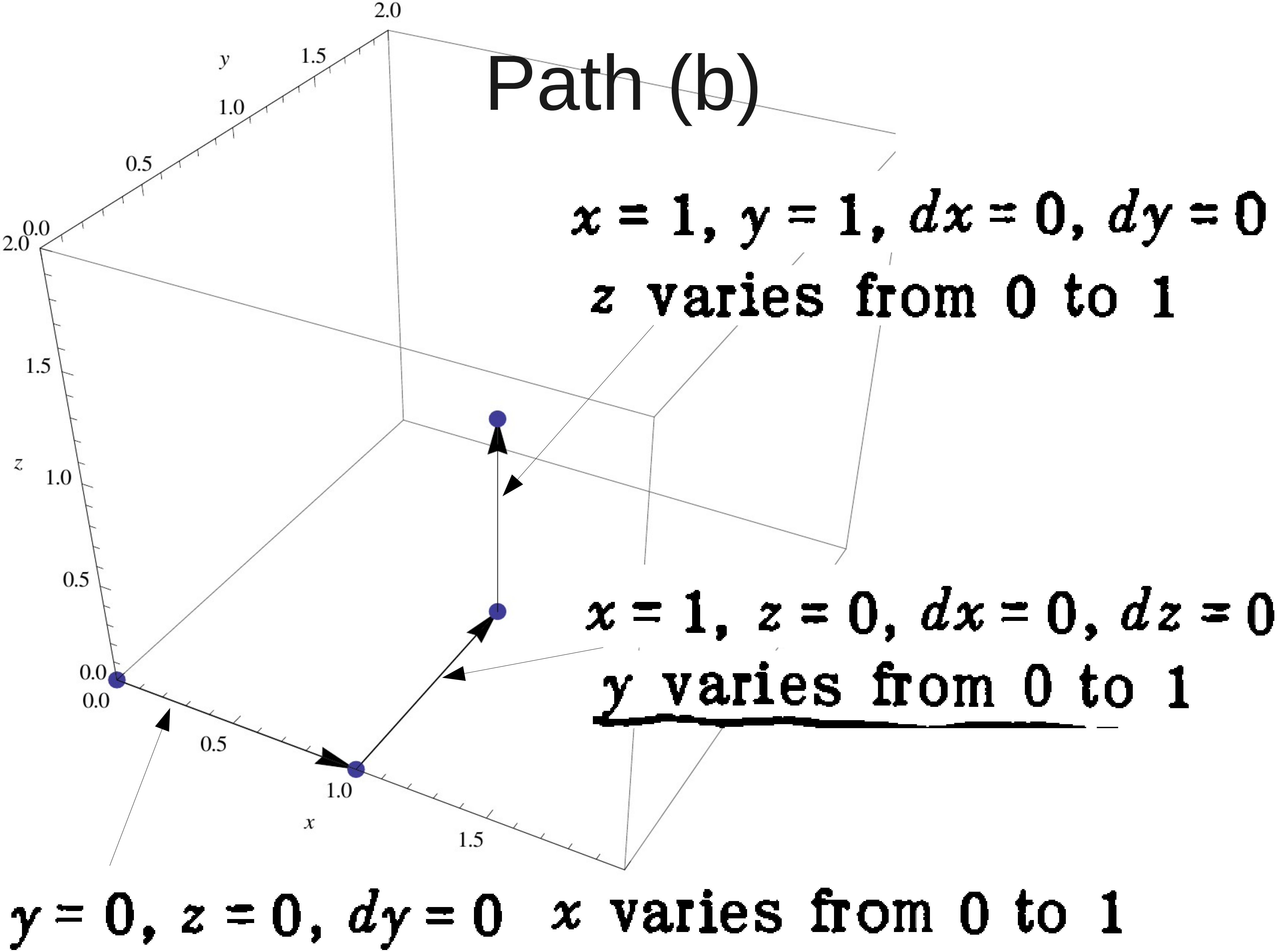
$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

6. If $\mathbf{A} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A} \cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the follow-

(*b*) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

Path (b)


$x = 1,\ y = 1,\ dx = 0,\ dy = 0$
z varies from 0 to 1
$x = 1,\ z = 0,\ dx = 0,\ dz = 0$
y varies from 0 to 1
$y = 0,\ z = 0,\ dy = 0$
x varies from 0 to 1
x
y
z
0.0
0.5
1.0
1.5
2.0

The path *C* is divided into three segments, each is to be evaluated independently according to the geometry of the line segment involved.

$$\mathbf{A} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$$

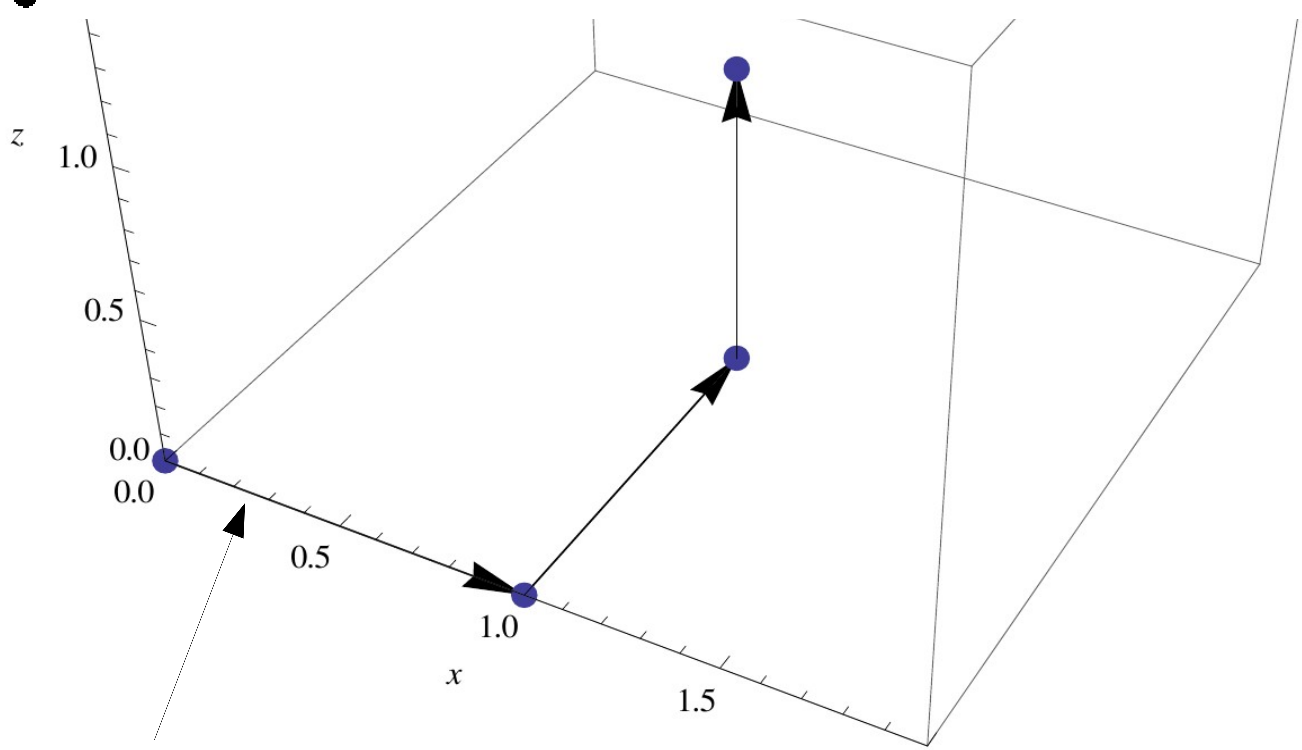
$$\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

$$\mathbf{A} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$$

Along the straight line from (0,0,0) to (1,0,0)

$$\int_{x=0}^{1} \left(3x^2 + 6(0)\right) dx - 14(0)(0)(0) + 20x(0)^2(0)$$

$$= 1$$

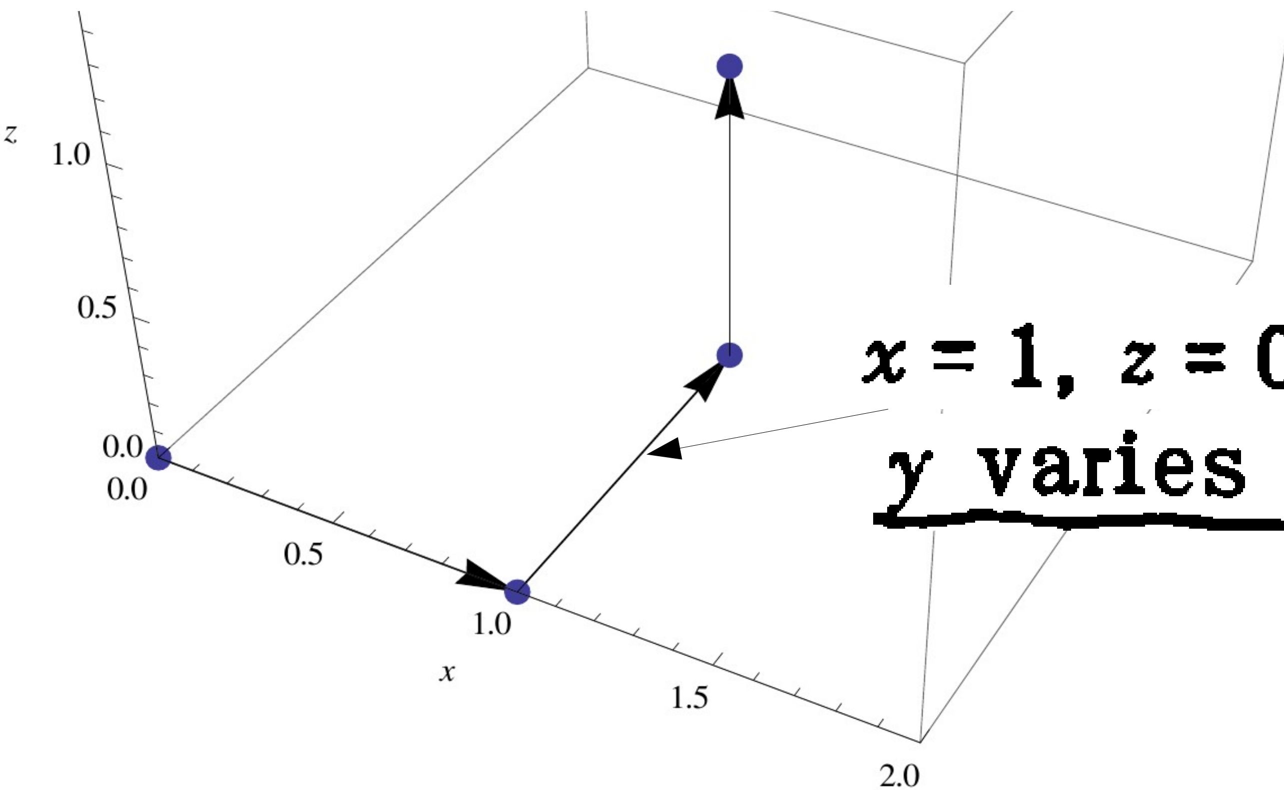


$y = 0,\ z = 0,\ dy = 0$ x varies from 0 to 1

$$\mathbf{A} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$$

Along the straight line from (1,0,0) to (1,1,0)

$$\int_{y=0}^{1} (3(1)^2 + 6y)\,0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$



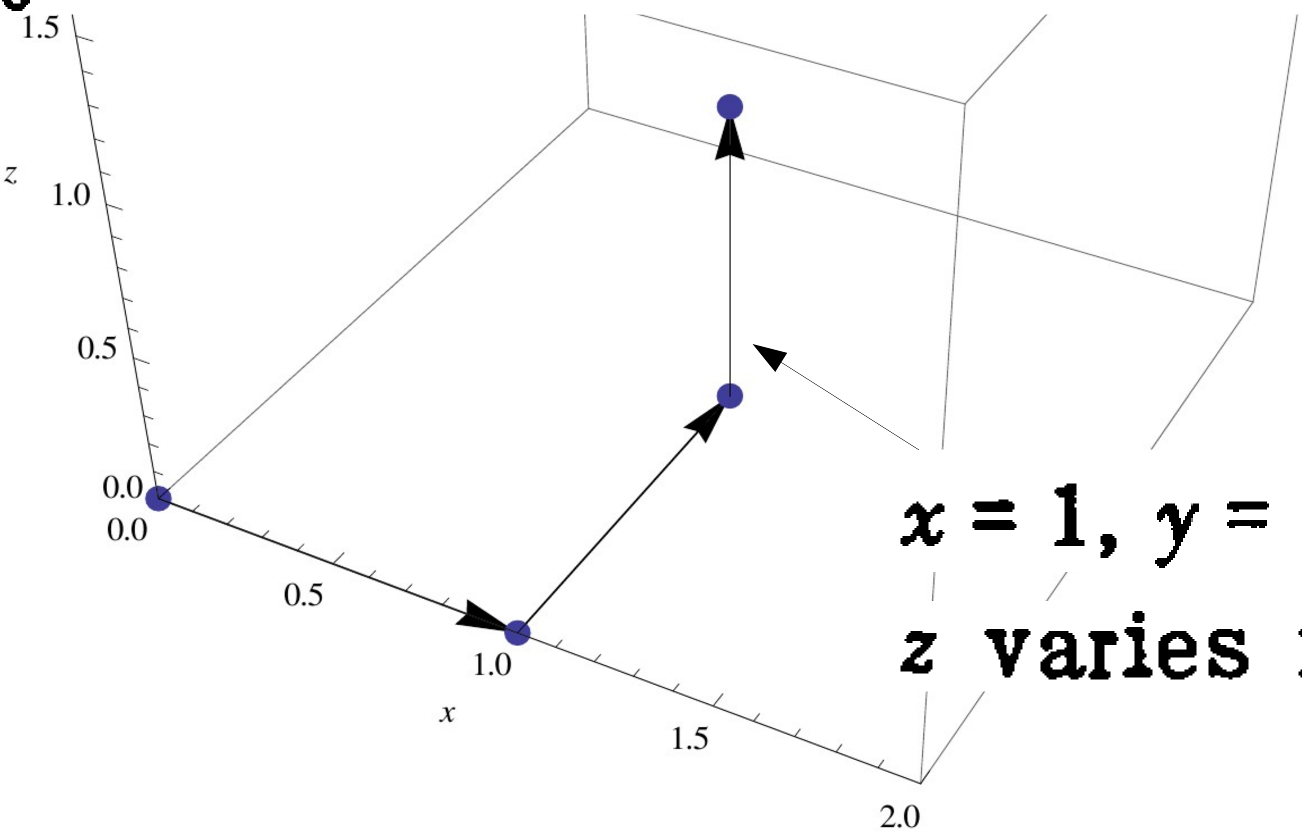
$x = 1,\ z = 0,\ dx = 0,\ dz = 0$

<u>y varies from 0 to 1</u>

$$\mathbf{A} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$$

Along the straight line from (1,1,0) to (1,1,1)

$$\int_{z=0}^{1} \left(3(1)^2 + 6(1)\right) 0 - 14(1)\,z(0) + 20(1)\,z^2\,dz = \frac{20}{3}$$



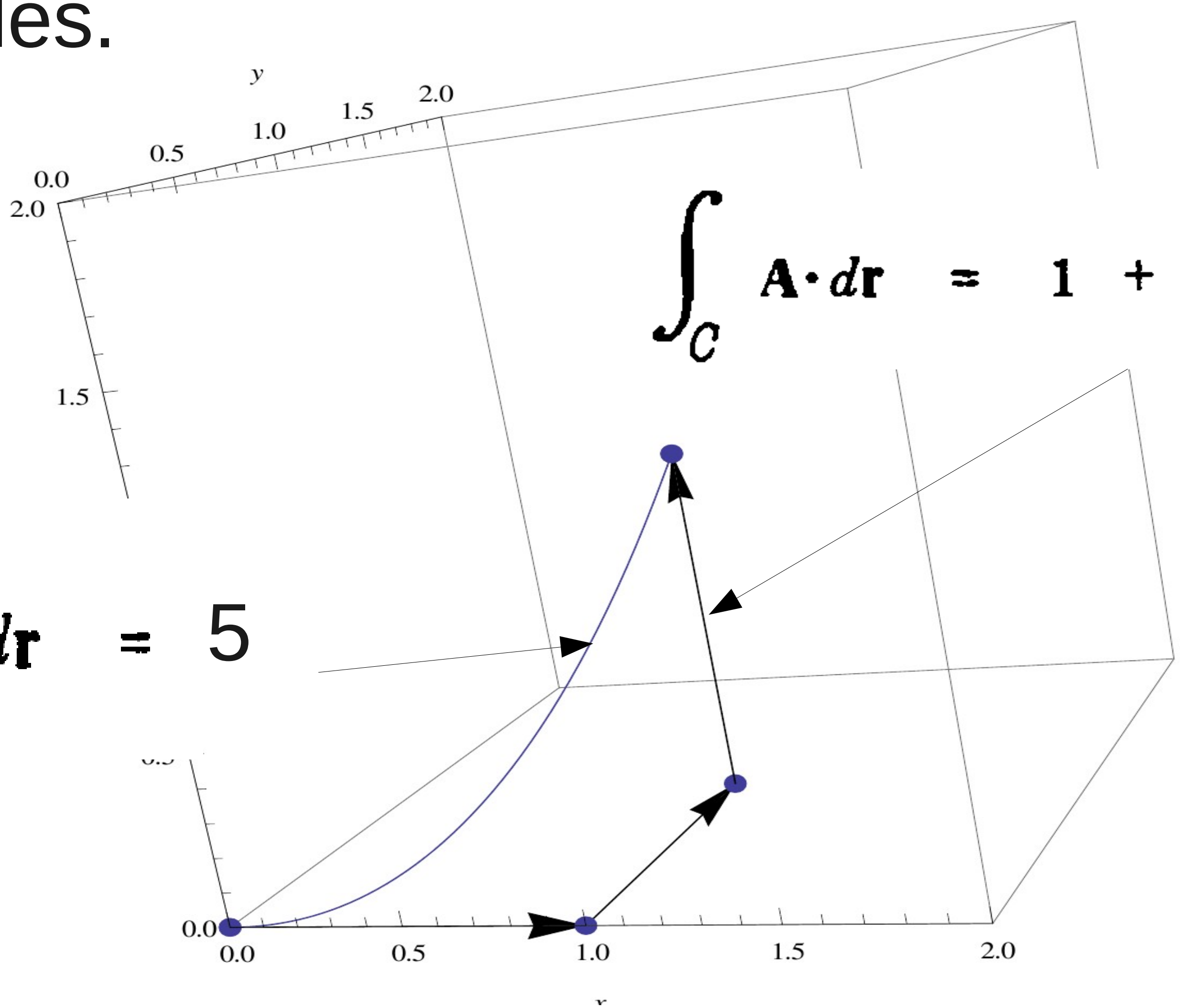
$x = 1,\ y = 1,\ dx = 0,\ dy = 0$

z varies from 0 to 1

Adding,

$$\int_C \mathbf{A}\cdot d\mathbf{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

Note: Line integration of a generic vector from one fixed point to another along two different paths are generally not the same, as seen in previous examples.

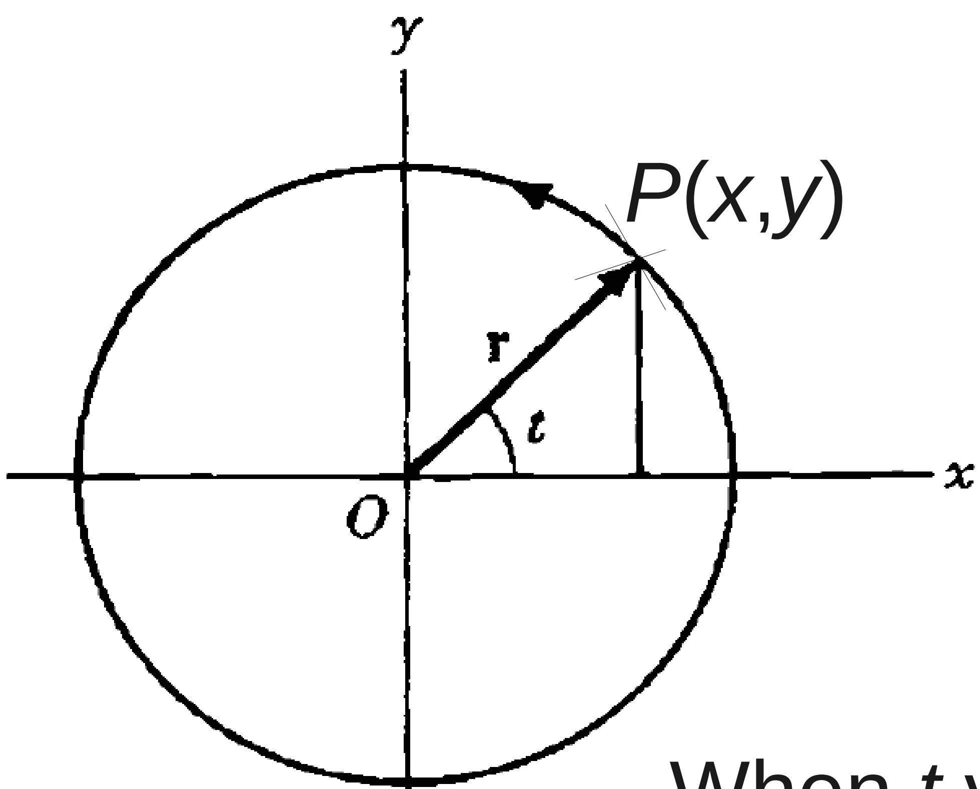


$$\int_C \mathbf{A} \cdot d\mathbf{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

$$\int_C \mathbf{A} \cdot d\mathbf{r} = 5$$

9. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$$



The circular path with radius r is parametrised by the parametric equations

$x = \cos t$, $y = \sin t$

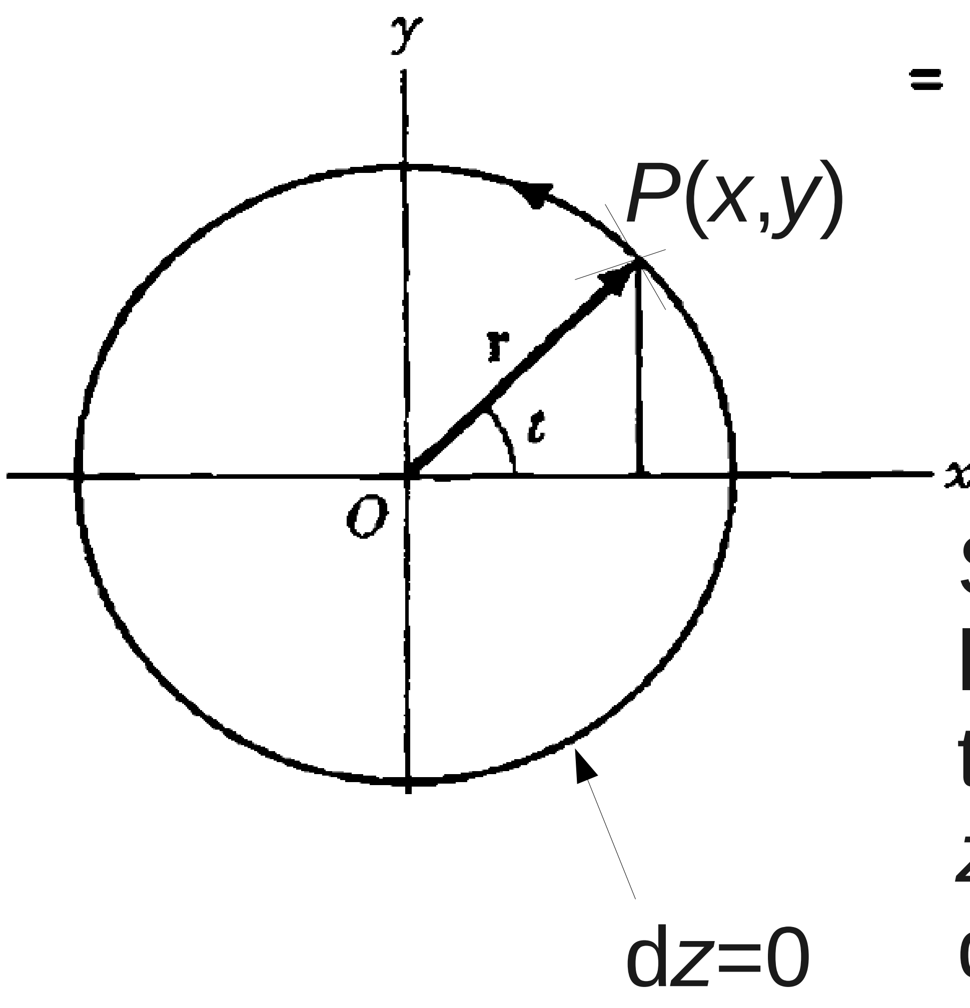
When t varies from 0 to 2π, point P would have traversed clockwise by a full circle

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$$

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left[(2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}\right]\cdot\left[dx\,\mathbf{i} + dy\,\mathbf{j}\right]$$

$$= \int_C (2x-y)\,dx + (x+y)\,dy$$



Since the circular path *C* is located on the x-y plane, there is no variation in the *z*-variable, hence d*z*=0 in d**r**.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (2x-y)\,dx + (x+y)\,dy$$

x=cos *t*, d*x* =-sin *t* d*t*; *y* = sin *t*, *dy*=cos *t* d*t*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{2\pi} \Big\{ [2(3\cos t) - 3\sin t]\,[-3\sin t]\,dt + [3\cos t + 3\sin t]\,[3\cos t]\,dt \Big\}$$

$$= 18\pi$$

Setting the limit of integration from *t*=0 to *t*=2π amounts to performing the line integration in the anticlockwise (+ ve) direction.

If the limit of integration is instead set from t=2π to t=0, that amounts to performing the line integration in the clockwise (-ve) direction. As result, the line integration would result in a relative negative sign:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=2\pi}^{0} \left\{ [2(3\cos t) - 3\sin t][-3\sin t]\,dt + [3\cos t + 3\sin t][3\cos t]\,dt \right\}$$

$$= -18\pi$$

Clockwise , -ve direction

Line integration is direction-dependent

From the previous example, it is seen that in a given line integration, the direction of integration matters.

In most cases, integrating along opposite direction results in a relative minus sign between two line integrations (but this is not in general true).

Line integration is direction-dependent

So, which is the correct answer in the previous example? The +ve one or the -ve one?

The correct statements are:

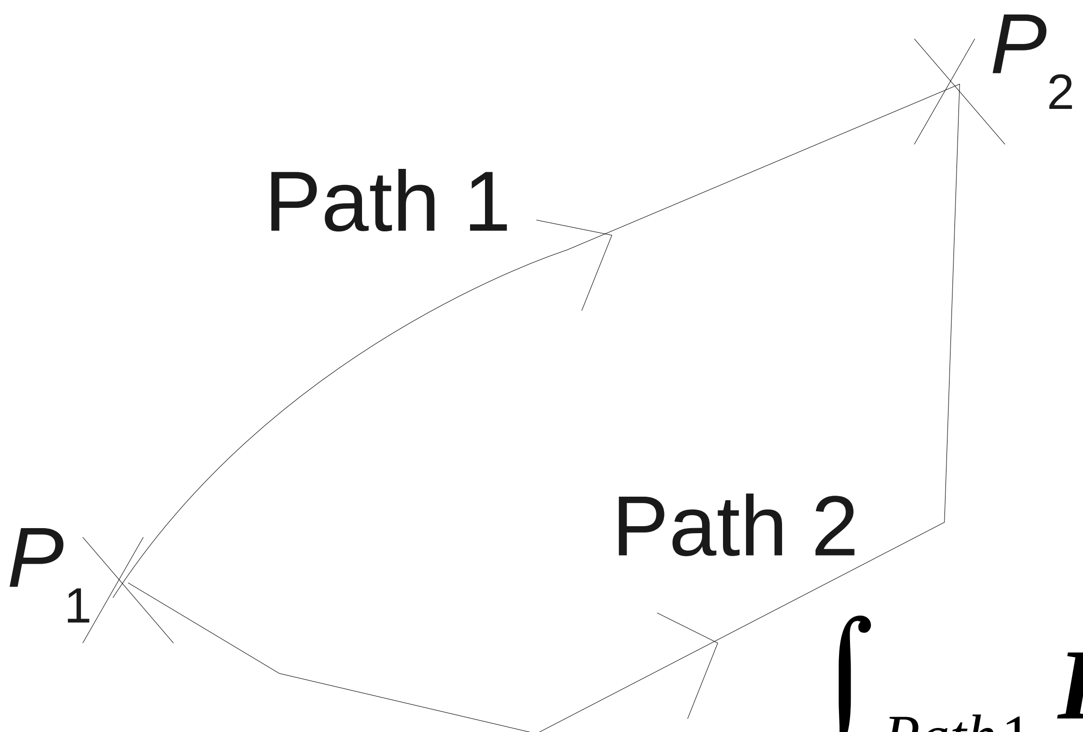
The work done is $+18\pi$ if integrate along the circle in the clockwise direction;

The work done is -18π if integrate along the circle in the anticlockwise direction;

10. (*a*) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

(*b*) Conversely, if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $\mathbf{F} = \nabla\phi$.

10. (a) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.



You are asked to show that, if

$$\boldsymbol{F} = \nabla\phi$$

then

$$\int_{Path1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{Path2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

If $\boldsymbol{F} = \nabla \phi$

Work done

$$= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_2} \nabla\phi \cdot d\mathbf{r}$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \right) \cdot \left(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} \right)$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them.

Conservative field

If $\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining P_1 and P_2, then $\mathbf{F}$ is called a *conservative field*.

Conversely, we can also show that if the line integration

$$\int_C \boldsymbol{F} \cdot d\,\boldsymbol{r}$$

is independent of the path *C* joining any two points, there exist a function ϕ such that

$$\boldsymbol{F} = \nabla \phi$$

ϕ is known as the 'scalar potential'

Proof using vectors.

If the line integral is independent of the path, then

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \mathbf{F}\cdot \frac{d\mathbf{r}}{ds}\, ds$$

By differentiation, $\dfrac{d\phi}{ds} = \mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}$.

But $\dfrac{d\phi}{ds} = \nabla\phi\cdot\dfrac{d\mathbf{r}}{ds}$ (see next slide for a proof)

$$(\nabla\phi - \mathbf{F})\cdot\frac{d\mathbf{r}}{ds} = 0$$

Since this must hold irrespective of $\dfrac{d\mathbf{r}}{ds}$, we have $\mathbf{F} = \nabla\phi$.

$$\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\mathbf{r}}{ds}$$

Chap 4, page 61, Q8, part (b), (c)

(*b*) Evaluate $\lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = \frac{d\phi}{ds}$

(*c*) Show that $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\mathbf{r}}{ds}$.

From the calculus,

$$\Delta\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z$$

$$\lim_{\Delta s \to 0}\frac{\Delta\phi}{\Delta s} = \lim_{\Delta s \to 0}\frac{\partial\phi}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial\phi}{\partial y}\frac{\Delta y}{\Delta s} + \frac{\partial\phi}{\partial z}\frac{\Delta z}{\Delta s}$$

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)$$

$$= \nabla\phi\cdot\frac{d\mathbf{r}}{ds}$$

$$\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\mathbf{r}}{ds}$$

Note that since $\frac{d\mathbf{r}}{ds}$ is a unit vector, $\nabla\phi \cdot \frac{d\mathbf{r}}{ds}$ is the component of $\nabla\phi$ in the direction of this unit vector.

Conclusion of Q10

A conservative field is one which line integration of a field is independent of path.

If a field can be expressed in terms of a differentiable scalar function ϕ via

$$\boldsymbol{A} = \nabla \phi$$

then **A** is conservative.

Conversely, if a field **A** is conservative, it can always be expressed a gradient of some scalar function,

$$\boldsymbol{A} = \nabla \phi$$

11.

(a) If $\mathbf{F}$ is a conservative field, prove that $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational).

If $\mathbf{F}$ is a conservative field, then by Problem 10, $\mathbf{F} = \nabla \phi$.

$$\operatorname{curl} \mathbf{F} = \nabla \times \nabla \phi = \mathbf{0}$$

(see Problem 27(a), Chapter 4).

11.

(*b*) Conversely, if $\nabla \times \mathbf{F} = \mathbf{0}$ (i.e. **F** is irrotational), prove that **F** is conservative.

If $\nabla \times \mathbf{F} = \mathbf{0}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{0}$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

11.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

it can be proven that

$\mathbf{F} = \nabla\phi$ follows as a consequence of this.

For details of the proof, read it yourself.

Thus a necessary and sufficient condition that a field $\mathbf{F}$ be conservative is that $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$.

THEOREM. If $\mathbf{A} = \nabla\phi$ then

1. $\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r}$ is independent of the path C in R joining P_1 and P_2.

(i.e., **A** is conservative)

2. $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$ around any closed curve C in R.

Theorem No.1 is what we have just proven in Q10.

Theorem No.2 can be proved using Theorem 1.

Closed Line integration

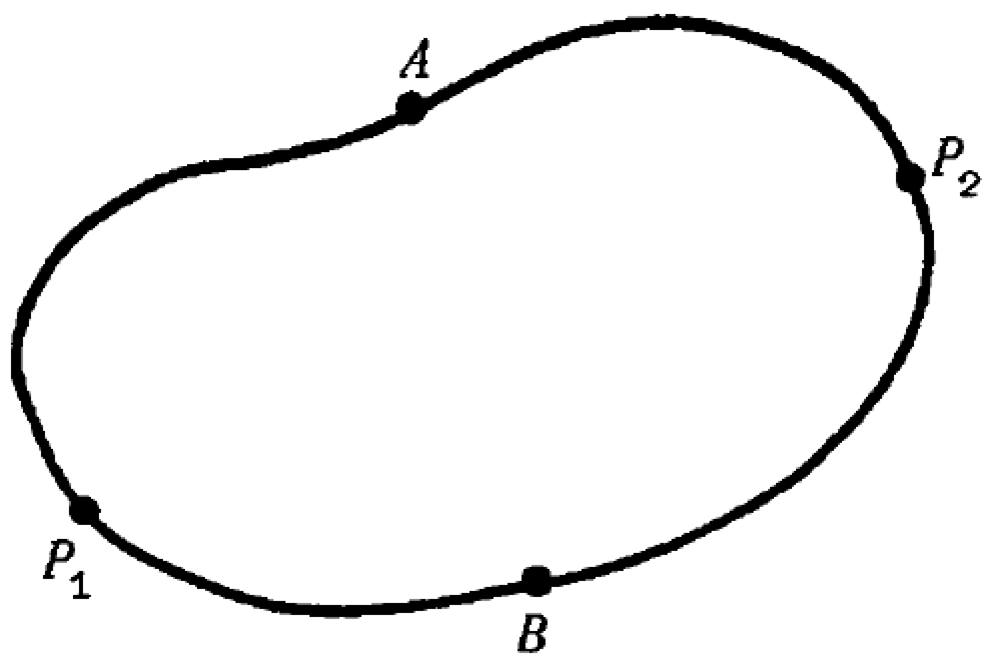
Consider a line integration along a path *C*

$$\int_C \mathbf{A} \cdot d\mathbf{r}$$

If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A} \cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

13. Prove that if $\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining any any two points P_1 and P_2 in a given region, then $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths in the region and conversely.



Need to prove two statements:

If $\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining any two points P_1 and P_2

then $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths

---

If $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths in the region

then

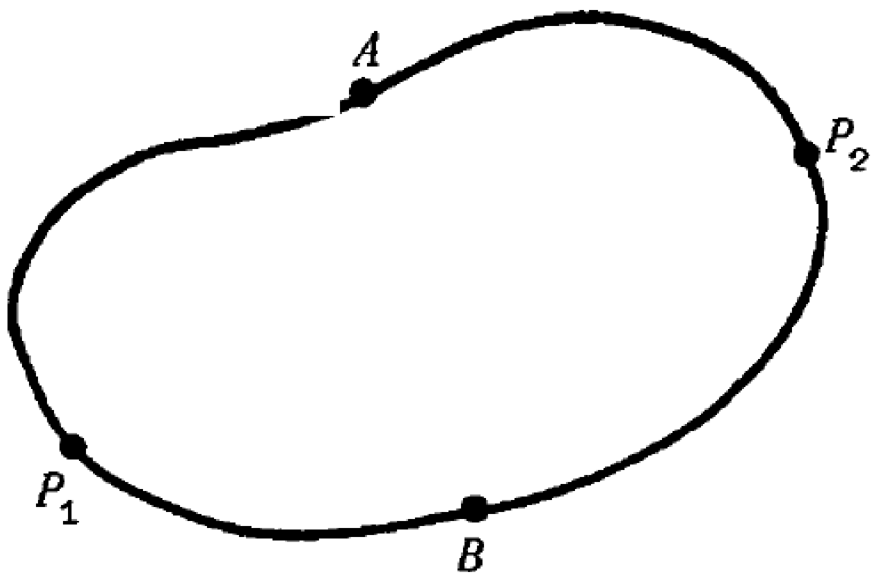
$\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining any two points P_1 and P_2

Proof of the first statement

Let $P_1AP_2BP_1$ be a closed curve. Then

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} - \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} = 0$$



The first part is proven. But we still need to prove the converse statement.

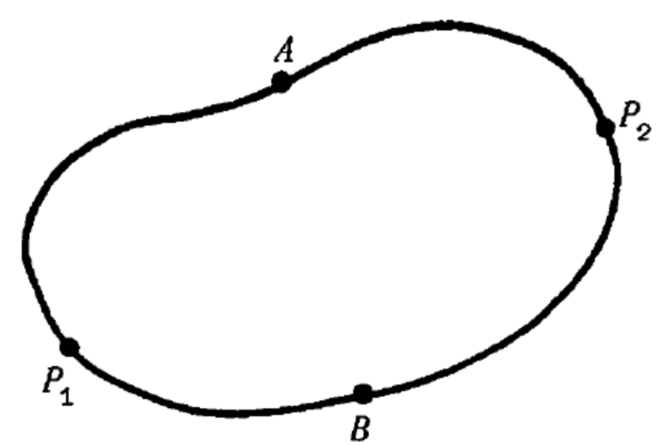
Proof of the converse statement

if $\oint \mathbf{F}\cdot d\mathbf{r} = 0$, then

$$\int_{P_1AP_2BP_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2BP_1} \mathbf{F}\cdot d\mathbf{r} =$$

$$\int_{P_1AP_2} \mathbf{F}\cdot d\mathbf{r} - \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

$$\int_{P_1AP_2} \mathbf{F}\cdot dr = \int_{P_1BP_2} \mathbf{F}\cdot d\mathbf{r}$$



Conclusion from Q10, Q11, Q13

The following statements are equivalent:

F is conservative

$$\mathbf{F} = \nabla\phi$$

$$\nabla\times\mathbf{F} = \mathbf{0}$$

$\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points

$\oint \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed paths

12.

(*a*) Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is a conservative force field.

(*b*) Find the scalar potential.

(*c*) Find the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

12.

(a) Show that $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ is a conservative force field.

Easiest to prove the statement by showing that the curl of **F** is zero.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \mathbf{0}.$$

Thus **F** is a conservative force field.

(*b*) Find the scalar potential.

First Method.

$$\mathbf{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}.$$

(1) $\dfrac{\partial\phi}{\partial x} = 2xy + z^3$ (2) $\dfrac{\partial\phi}{\partial y} = x^2$ (3) $\dfrac{\partial\phi}{\partial z} = 3xz^2$

Integrate (1), (2) and (3) to obtain ϕ

(b) Find the scalar potential.

(1) $\frac{\partial\phi}{\partial x} = 2xy + z^3$ (2) $\frac{\partial\phi}{\partial y} = x^2$ (3) $\frac{\partial\phi}{\partial z} = 3xz^2$

$$\phi = x^2y + xz^3 + f(y,z)$$

$$\phi = x^2y + g(x,z)$$

$$\phi = xz^3 + h(x,y)$$

These agree if we choose $f(y,z) = 0$, $g(x,z) = xz^3$, $h(x,y) = x^2y$

so that $\phi = x^2y + xz^3$ + constant

(b) Find the scalar potential.

Another method

$$\mathbf{F} \cdot d\mathbf{r} = \nabla\phi \cdot d\mathbf{r}$$

$$= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$$

$$d\phi = \mathbf{F} \cdot d\mathbf{r}$$

$$= (2xy + z^3)\, dx + x^2\, dy + 3xz^2\, dz$$

$$= (2xy\, dx + x^2\, dy) + (z^3\, dx + 3xz^2\, dz)$$

$$= d(x^2 y) + d(xz^3) = d(x^2 y + xz^3)$$

$$\phi = x^2 y + xz^3 + \text{constant.}$$

(c) Work done $= \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{P_1}^{P_2} (2xy + z^3)\, dx + x^2\, dy + 3xz^2\, dz$$

$$= \int_{P_1}^{P_2} d(x^2 y + xz^3) = x^2 y + xz^3 \Big|_{P_1}^{P_2}$$

$$= x^2 y + xz^3 \Big|_{(1,-2,1)}^{(3,1,4)} = 202$$

$$(c)\ \text{Work done} = \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r}$$

A better still method is to use the potential function

$$\phi = x^2y + xz^3 + \text{constant.}$$

$$\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r}$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$d\phi = \mathbf{F}\cdot d\mathbf{r}$$

$$\text{Work done} = \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \Delta\phi$$

$$= \phi(3,1,4) - \phi(1,-2,1) = 202$$

Work done by conservative field and potential energy difference

For conservative fields, work done by the field when moving a particle from point P_1 to P_2 is equal to the change of potential energy between P_1 to P_2

$$W(\boldsymbol{r}_1 \rightarrow \boldsymbol{r}_2) = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\,\boldsymbol{r} = \Delta\phi = \phi(\boldsymbol{r}_2) - \phi(\boldsymbol{r}_1)$$

Have you seen any conservative field before?

Of course yes.

Gravitational force, $\boldsymbol{F}_g(\boldsymbol{r}) = -\frac{GMm}{r^2}\hat{r}$

Its corresponding potential function, $U(\boldsymbol{r}) = -\frac{GMm}{r}$

Both are related via $\boldsymbol{F}_g(\boldsymbol{r}) = \nabla U(\boldsymbol{r})$

Work done by gravitational force on a mass *m* from $\mathbf{r}_1$ to $\mathbf{r}_2$

$$W(\boldsymbol{r}_1 \rightarrow \boldsymbol{r}_2) = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_g(\boldsymbol{r}) \cdot d\,\boldsymbol{r} = \Delta U = U(\boldsymbol{r}_2) - U(\boldsymbol{r}_1)$$

Other conservative fields you have seen before

Electrostatic field is also a conservative field.

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{qQ}{4\pi\epsilon r^2}\hat{r} \qquad U(\boldsymbol{r}) = \frac{Qq}{4\pi\epsilon r} \qquad \boldsymbol{F}(\boldsymbol{r}) = \nabla U(\boldsymbol{r})$$

Elastic force of a spring,

$$\boldsymbol{F}(x) = k\hat{x}x \qquad U(x) = \frac{k}{2}x^2 \qquad \boldsymbol{F}(x) = \frac{d}{dx}U(x)$$

What else?

Is frictional force conservative?

- No.
- Why?
- Despite the end points are the same, work done by frictional force is path dependent.
- Hence, there is no corresponding “frictional potential” from which work done can be calculated as the difference of the “potential function” between two points.

ZCT 211 Vector Analysis

Tutorial questions to submit. Take the questions from the textbook by Spiegel, Vector Analysis, Schaum series.

Chapter 1 Vectors and Scalar

Q38, Q39, Q44, Q47, Q59, Q61, Q64(a)

Q66(c) modified:

66. Graph the vector fields defined by

(c) $\mathbf{V}(x, y\quad) = \dfrac{x\mathbf{i} + y\mathbf{i}}{\sqrt{x^2 + y^2}}$

Tutorial for Chapter 2

64. Find the projection of the vector $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ on the line passing through the points $(2,3,-1)$ and $(-2,-4,3)$.

65. If $\mathbf{A} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

80. If $\mathbf{A} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{C} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, find:

(a) $|(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}|$, (c) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, (e) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{B} \times \mathbf{C})$

(b) $|\mathbf{A} \times (\mathbf{B} \times \mathbf{C})|$, (d) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, (f) $(\mathbf{A} \times \mathbf{B})(\mathbf{B} \cdot \mathbf{C})$

83. Find the area of a triangle with vertices at $(3,-1,2)$, $(1,-1,-3)$ and $(4,-3,1)$.

88. Simplify $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{B} + \mathbf{C}) \times (\mathbf{C} + \mathbf{A})$.

95. Let points P, Q and R have position vectors $\mathbf{r}_1 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{r}_3 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ relative to an origin O. Find the distance from P to the plane OQR.

100. Prove that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) + (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{A} \times \mathbf{D}) + (\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{B} \times \mathbf{D}) = 0$.

Tutorial 3

Pass up personally
on
Monday class
12 pm, 27 Oct 2014

Tutorial for Chapter 3

32.

Find the velocity and acceleration of a particle which moves along the curve $x = 2 \sin 3t$, $y = 2 \cos 3t$, $z = 8t$ at any time $t > 0$. Find the magnitude of the velocity and acceleration.

Ans. $\mathbf{v} = 6 \cos 3t\, \mathbf{i} - 6 \sin 3t\, \mathbf{j} + 8\mathbf{k}$, $\mathbf{a} = -18 \sin 3t\, \mathbf{i} - 18 \cos 3t\, \mathbf{j}$, $|\mathbf{v}| = 10$, $|\mathbf{a}| = 18$

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34. If $\mathbf{A} = t^2 \mathbf{i} - t\mathbf{j} + (2t+1)\mathbf{k}$ and $\mathbf{B} = (2t-3)\mathbf{i} + \mathbf{j} - t\mathbf{k}$, find (*a*) $\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B})$, (*b*) $\frac{d}{dt}(\mathbf{A} \times \mathbf{B})$, (*c*) $\frac{d}{dt}|\mathbf{A} + \mathbf{B}|$, (*d*) $\frac{d}{dt}(\mathbf{A} \times \frac{d\mathbf{B}}{dt})$ at $t = 1$.

Ans. (*a*) -6, (*b*) $7\mathbf{j} + 3\mathbf{k}$, (*c*) 1,
(*d*) $\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$

38.

If $\frac{d^2\mathbf{A}}{dt^2} = 6t\, \mathbf{i} - 24t^2\, \mathbf{j} + 4 \sin t\, \mathbf{k}$, find $\mathbf{A}$ given that $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$ and $\frac{d\mathbf{A}}{dt} = -\mathbf{i} - 3\mathbf{k}$ at $t = 0$.

Ans. $\mathbf{A} = (t^3 - t + 2)\mathbf{i} + (1 - 2t^4)\mathbf{j} + (t - 4 \sin t)\mathbf{k}$

44.

If $\mathbf{A} = x^2yz\, \mathbf{i} - 2xz^3\, \mathbf{j} + xz^2\, \mathbf{k}$ and $\mathbf{B} = 2z\, \mathbf{i} + y\, \mathbf{j} - x^2\, \mathbf{k}$, find $\frac{\partial}{\partial x\, \partial y}(\mathbf{A} \times \mathbf{B})$ at $(1, 0, -2)$.

Ans. $-4\mathbf{i} - 8\mathbf{j}$

45.

If $\mathbf{C}_1$ and $\mathbf{C}_2$ are constant vectors and λ is a constant scalar, show that $\mathbf{H} = e^{-\lambda x}(\mathbf{C}_1 \sin \lambda y + \mathbf{C}_2 \cos \lambda y)$ satisfies the partial differential equation $\dfrac{\partial^2 \mathbf{H}}{\partial x^2} + \dfrac{\partial^2 \mathbf{H}}{\partial y^2} = \mathbf{0}$.

47.

Find (*a*) the unit tangent $\mathbf{T}$, (*b*) the curvature κ, (*c*) the principal normal $\mathbf{N}$, (*d*) the binormal $\mathbf{B}$, and (*e*) the torsion τ for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$.

Ans. (*a*) $\mathbf{T} = \dfrac{(1-t^2)\mathbf{i} + 2t\,\mathbf{j} + (1+t^2)\mathbf{k}}{\sqrt{2}(1+t^2)}$ (*c*) $\mathbf{N} = -\dfrac{2t}{1+t^2}\mathbf{i} + \dfrac{1-t^2}{1+t^2}\mathbf{j}$ (*e*) $\tau = \dfrac{1}{(1+t^2)^2}$

(*b*) $\kappa = \dfrac{1}{(1+t^2)^2}$ (*d*) $\mathbf{B} = \dfrac{(t^2-1)\mathbf{i} - 2t\,\mathbf{j} + (t^2+1)\mathbf{k}}{\sqrt{2}(1+t^2)}$

66.

A particle moves along the curve $\mathbf{r} = (t^3 - 4t)\mathbf{i} + (t^2 + 4t)\mathbf{j} + (8t^2 - 3t^3)\mathbf{k}$, where t is the time. Find the magnitudes of the tangential and normal components of its acceleration when $t = 2$.

Ans. Tangential, 16 ; normal, $2\sqrt{73}$

Tutorial 4

Pass up personally
on
Friday 10 pm class
7 Nov 2011

Tutorial for Chapter 4

44. If $F = x^2 z + e^{y/x}$ and $G = 2z^2 y - xy^2$, find (*a*) $\nabla(F+G)$ and (*b*) $\nabla(FG)$ at the point $(1,0,-2)$.
Ans. (*a*) $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, (*b*) $-8\mathbf{j}$

46. Prove $\nabla f(r) = \dfrac{f'(r)\,\mathbf{r}}{r}$.

51. If $\nabla\phi = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}$, find $\phi(x,y,z)$ if $\phi(1,-2,2) = 4$. *Ans.* $\phi = x^2yz^3 + 20$

54. If F is a differentiable function of x,y,z,t where x,y,z are differentiable functions of t, prove that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla F \cdot \frac{d\mathbf{r}}{dt}$$

55. If $\mathbf{A}$ is a constant vector, prove $\nabla(\mathbf{r}\cdot\mathbf{A}) = \mathbf{A}$.

58. Find a unit vector which is perpendicular to the surface of the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point (1,2,5).

Ans. $\dfrac{2\mathbf{i} + 4\mathbf{j} - \mathbf{k}}{\pm\sqrt{21}}$

64. In what direction from the point (1,3,2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

Ans. In the direction of the vector $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, $2\sqrt{14}$

76. If $\boldsymbol{\omega}$ is a constant vector and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove that $\operatorname{div}\mathbf{v} = 0$.

82. If $\mathbf{A} = \mathbf{r}/r$, find grad div $\mathbf{A}$. *Ans.* $-2r^{-3}\,\mathbf{r}$

91. Evaluate $\nabla \times (\mathbf{r}/r^2)$. *Ans.* $\mathbf{0}$

103. Show that $\mathbf{E} = \mathbf{r}/r^2$ is irrotational. Find ϕ such that $\mathbf{E} = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$.

Ans. $\phi = \ln(a/r)$

104. If $\mathbf{A}$ and $\mathbf{B}$ are irrotational, prove that $\mathbf{A} \times \mathbf{B}$ is solenoidal.

105. If $f(r)$ is differentiable, prove that $f(r)\mathbf{r}$ is irrotational.